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A Spinor Representation of Maxwell's Equations (*).

H. E. MOSES

Institute of Mathematical Sciences, New York University - New York

(ricevuto il 16 Marzo 1957)

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1. - Introduction.

In order to display the relativistic form of Maxwell's equations, one usually writes them either in the familiar tensor form or the spinor form of LAPORTE and UHLENBECK [1].

OPPENHEIMER [2] was able to cast Maxwell's equations in a form very similar to Dirac's equations. He analyzed the transformation properties by appealing to the usual two-component spinor formalism.

In the present work we shall obtain a form for Maxwell's equations which are *identical* to Dirac's equations for particles of mass zero and which have

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(1) O. LAPORTE and G. E. UHLENBECK: *Phys. Rev.*, **37**, 1380, 1552 (1931).

(2) J. R. OPPENHEIMER: *Phys. Rev.*, **38**, 725 (1931).

sources ([3,4]). Rather than using the powerful but somewhat complicated spinor formalism to analyze the transformation properties, we shall use a simpler method which is analogous to that used by DIRAC ([5]). We shall show that the requirement that one component of the spinor be an invariant leads to transformations for the electromagnetic field different from those for the particles of spin $\frac{1}{2}$.

The notation which we use is motivated by that used by MOLIÈRE, who uses a truncated set of Maxwell's equations for another purpose ([6]).

The fact that the Dirac operator is the square root of the wave operator will enable us to obtain in a simple manner several results usually obtained through the use of vector analysis. Furthermore, it will be easy to introduce vector potentials with the aid of spinors and to give some of tehir properties. We shall also give formulas by which one can obtain a vector potential from a knowledge of the field strengths and shall show, in the case where there are no sources, how the anti-potential ([7]) (which is the potential one can introduce by interchanging the roles of the electric and magnetic fields) is related to the conventional vector potential.

2. – The Dirac form for Maxwell's equations.

We introduce the four component vectors

$$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix}, \tag{2.1}$$

([3]) The present paper is a rewrite of the report by H. E. MOSES: *The Spinor Transformation of Maxwell's Equations*, Report No. IMM-NYU 238, January, 1957, New York University, Institute of Mathematical Sciences, New York. The author's attention has been drawn to reference ([4]) which has appeared in the meantime, where substantially the same form of Maxwell's equations are given as in the present paper. However, the transformation properties are not discussed.

([4]) T. OHMURA: *Prog. Theor. Phys.*, **16**, 604 (1956).

([5]) P. A. M. DIRAC: *The Principles of Quantum Mechanics*, third edition (Oxford, 1947), Chapter XI.

([6]) G. MOLIÈRE: *Ann. der Phys.*, **6**, 146 (1949).

([7]) This word is used by M. BORN as quoted in B. B. BAKER and E. T. COPSON: *The Mathematical Theory of Huygen's Principle* (Oxford, 1939), p. 105.

where

$$(2.1a)\qquad \begin{cases} \psi_0 = 0 & \Phi_0 = \varrho \\ \psi_1 = H_x - iE_x & \Phi_1 = j_x \\ \psi_2 = H_y - iE_y & \Phi_2 = j_y \\ \psi_3 = H_z - iE_z & \Phi_3 = j_z \,. \end{cases}$$

The Maxwell's equations can be written

$$(2.2)\qquad -\frac{1}{i}\sum_{j=0}^{3}\alpha^j\frac{\partial}{\partial x^j}\psi = -\psi\pi\Phi \,.$$

In (2.2) α^i $(i = 0, 1, 2, 3)$ are matrices which act on ψ and are given by

$$(2.3)\qquad \begin{cases} \alpha^0 \equiv I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \\ \alpha^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \end{cases}$$

Furthermore, in (2.2) and later we write

$$(2.4)\qquad x^0 = x_0 = t\,, \quad x^1 = -x_1 = x\,, \quad x^2 = -x_2 = y\,, \quad x^3 = -x_3 = z\,.$$

It is clear that the matrices α^i are Hermitian and that

$$(2.5)\qquad \begin{cases} (\alpha^1)^2 = (\alpha^2)^2 = (\alpha^3)^2 = I\,, \\ \alpha^1\alpha^2 = i\alpha^3 = -\alpha^2\alpha^1\,, \\ \alpha^2\alpha^3 = i\alpha^1 = -\alpha^3\alpha^2\,, \\ \alpha^3\alpha^1 = i\alpha^2 = -\alpha^1\alpha^3\,. \end{cases}$$

Hence the matrices α^i satisfy the following anti-commutation rules

$$(2.6)\qquad \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}I\,, \qquad (i, j = 1, 2, 3)$$

where δ^{ij} is the usual Kronecker δ: $\delta^{ij} = 0$ for $i \neq j$, $\delta^{ii} = 1$. Thus the matrices α^i are a representation of the Dirac operators.

In particular, when there are no sources, Maxwell's equations can be written as

$$-\frac{1}{i}\frac{\partial}{\partial t}\psi = H_0\psi\,, \tag{2.7}$$

where H_0 is given by

$$H_0 = \frac{1}{i}\sum_{j=1}^{3}\alpha^j\frac{\partial}{\partial x^j}\,. \tag{2.8}$$

Hence, Maxwell's equations without sources are identical to Dirac's equations for a particle of mass zero. If one solved Dirac's equation for a particle of mass zero with initial conditions such that the top component of the spinor were identically zero for all time, one could interpret such a solution as a solution of Maxwell's equations.

Equations (2.6) lead to the usual result that the Dirac operator is the square root of the wave operator:

$$\left(-\frac{1}{i}\sum_{j=0}^{3}\alpha^j\frac{\partial}{\partial x_j}\right)\left(-\frac{1}{i}\sum_{k=0}^{3}\alpha^k\frac{\partial}{\partial x^k}\right) = \tag{2.9}$$

$$= \left(-\frac{1}{i}\sum_{k=0}^{3}\alpha^k\frac{\partial}{\partial x^k}\right)\left(-\frac{1}{i}\sum_{j=0}^{3}\alpha^j\frac{\partial}{\partial x_j}\right) = \Box^2 = \left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right).$$

On applying the operator $-(1/i)\sum_{j=0}^{3}\alpha^j(\partial/\partial x_j)$ to both sides of Maxwell's equations (2.2), one obtains

$$\Box^2\psi = \frac{\psi\pi}{i}\sum_{j=0}^{3}\alpha^j\frac{\partial}{\partial x_j}\Phi\,. \tag{2.10}$$

By identifying the components on both sides of (2.10) and using $\psi_0 = 0$, one immediately obtains the equation of continuity

$$\sum_{j=0}^{3}\frac{\partial\Phi_j}{\partial x^j} = \frac{\partial\varrho}{\partial t} + \operatorname{div}\boldsymbol{j} = 0 \tag{2.11}$$

as a necessary condition that Maxwell's equations have a solution. One also sees that if there are no sources, the electric and magnetic field components satisfy the wave equation separately.

3. – The transformation properties of the fields and sources under a Lorentz transformation. Spin.

We shall now discuss how ψ and Φ transform under a proper Lorentz transformation and why the Maxwell spinor transforms differently than the usual Dirac spinor.

Let us introduce the Lorentz transformation.

$$x^{i'} = \sum_{j=0}^{3} a^{i}{}_{j} x^{j}, \qquad (i = 0, 1, 2, 3). \tag{3.1}$$

Then Maxwell's equations (2.2) take the form

$$-\frac{1}{i} \sum_{j=0}^{3} \alpha^{j'} \frac{\partial}{\partial x^{j'}} \psi = -4\pi\Phi, \tag{3.2}$$

where

$$\alpha^{i'} = \sum_{j=0}^{3} a^{i}{}_{j} \alpha^{j}. \tag{3.2a}$$

In order that Maxwell's equations be invariant we must look for a ψ' and a Φ' which are related to ψ and Φ such that (3.2) leads to

$$-\frac{1}{i} \sum_{j=0}^{3} \alpha^{j} \frac{\partial}{\partial x^{j}} \psi' = \Phi'. \tag{3.3}$$

It is important to note that we must have

$$\psi'_0 \equiv 0. \tag{3.3a}$$

Let us make the Ansatz that ψ transforms as

$$\psi' = S\psi, \tag{3.4}$$

where S is a matrix,

$$S = (S_{ij}), \qquad (i, j = 0, 1, 2, 3). \tag{3.4a}$$

Since there is no *a priori* reason to expect Φ (or equivalently $\sum_{j=0}^{3} \alpha^{j}(\partial/\partial x^{j})\psi$) to transform in the same way as ψ, we assume a different transformation law for Φ (or $\sum_{j=0}^{3} \alpha^{j}(\partial/\partial x^{j})\psi$), namely

$$\Phi' = T\Phi, \tag{3.5}$$

where T is a matrix

$$T = (T_{ij}) , \qquad (i, j = 0, 1, 2, 3). \tag{3.5a}$$

On substituting (3.5) and (3.4) into (3.3) and comparing the result with (3.2), we obtain the result

$$S = \alpha^i T \alpha^{i'} = \sum_{j=0}^{3} a^i{}_j \alpha^i T \alpha^j , \qquad (i = 0, 1, 2, 3), \tag{3.6}$$

as equations for S and T. It is to be noted that (3.6) constitutes *four* equations, all of which must be satisfied by the same matrices S and T.

Equations (3.6) are not sufficient to guarantee the invariance of Maxwell's equations, for we must also use equation (3.3a) which states that the top component of the Maxwell spinor must be an invariant. This equation can be interpreted as a condition on S, namely

$$S_{01} = S_{02} = S_{03} = 0 . \tag{3.6a}$$

We are now in a position to see how our transformation differs from the usual Dirac transformation. Equation (3.6) also holds for the usual Dirac equation. However, instead of condition (3.6a), one has another condition involving the β matrix:

$$S = \beta T \beta , \tag{3.6b}$$

In the Dirac case (3.6b) leads to a solution for T in terms of S

$$T = \hat{S} , \tag{3.6c}$$

where the carat means Hermitian adjoint. On substituting (3.6c) into (3.6) one gets the usual S for the Dirac equation.

For the Maxwell case, however, we shall solve (3.6) for S and T under condition (3.6a). We obtain the following theorem:

Theorem: *Let the Lorentz transformation* (3.1) *be a proper one. Then equations* (3.6) *have only one solution for S and T under the condition* (3.6a), *the solution for T being given by*

$$T_{ij} = \lambda a^i{}_j , \tag{3.7}$$

where λ is an arbitrary scalar, and the same solution for S being given by any of the equations (3.6) *where T is given by* (3.7). *Furthermore, the matrices S/λ will form an orthogonal representation of the proper Lorentz group such that*

$$S_{00}/\lambda = 1 , \qquad S_{10} = S_{20} = S_{30} = 0 . \tag{3.8}$$

Before sketching the proof, we shall make some comments on the theorem, taking $\lambda = 1$. First of all, we note that the sources Φ transform like a four vector as in the usual theory. Secondly, since (3.6*a*) and (3.8) hold, we can see that four-by-four matrix S can be reduced to a one-dimensional matrix whose element is unity, and a three-by-three matrix which is orthogonal. Hence the matrices S form essentially a three-dimensional orthogonal representation of the Lorentz group. It should also be mentioned that on writing $\psi'_j = H'_j - iE'_j$, $(j = 1, 2, 3)$, it can be shown that the field strengths transform in the usual relativistic way. In Appendix I the matrix S is given explitiely in terms of $a^i{}_j$.

In particular, it should be noted that when the Lorentz transformation involves only a rotation in space; i.e. when

t3.9) $$a^0{}_0 = 1\,, \qquad a^0{}_i = 0\,, \qquad (i \neq 0),$$

(hen on using (3.6) with $i = 0$, we have

(3.10) $$S = T = (a^i{}_j)$$

and consequently the field strengths transform like an ordinary three-dimensional vector as they should.

The theorem can be proved in several ways with varying degrees of elegance. We shall sketch a simple proof.

Our proof will be based on two lemmas.

Lemma 1. *Any proper Lorentz transformation* (*a transformation such that* $a^0{}_0 > 0$, $\det a^i{}_j = 1$) *can be decomposed into a product of no more than six rotations, each rotation involving only two axes.*

A proof of Lemma 1 is given in Reference ([8]).

Lemma 2. *Consider a Lorentz transformation* L *which can be written as a product of other Lorentz transformations*

(3.11) $$L = \overset{1}{L}\overset{2}{L}\overset{3}{L} \ldots \overset{n}{L}$$

such that equations (3.6) *and* (3.6*a*) *have solutions for* $\overset{1}{L}, \overset{2}{L}, \ldots, \overset{n}{L}$, *the corresponding solutions for* S *being orthogonal matrices denoted by* $\overset{1}{S}, \overset{2}{S}, \ldots, \overset{n}{S}$, *and those for* T *by* $\overset{1}{T}, \overset{2}{T}, \ldots, \overset{n}{T}$. *Then equations* (3.6) *and* (3.6*a*) *have solutions for*

([8]) H. C. Lee: *Quart. Journ. Math.*, **15**, 7 (1944).

the transformation L, those solutions being given by

$$\text{(3.12)} \qquad \begin{cases} S = \overset{1}{S}\overset{2}{S}\dots\overset{n}{S}, \\ T = \overset{1}{T}\overset{2}{T}\dots\overset{n}{T}. \end{cases}$$

Proof: We shall prove the theorem for $n = 2$, since the proof can be extended in an obvious way for $n > 2$. Let $\overset{1}{a}{}^{i}{}_{j}$ and $\overset{2}{a}{}^{i}{}_{j}$ be the transformation coefficients for $\overset{1}{L}$ and $\overset{2}{L}$ respectively. Then (3.6) and (3.6a) have the form

$$\text{(3.13)} \qquad \begin{cases} \overset{1}{S} = \sum_{j} \overset{1}{a}{}^{i}{}_{j}\alpha^{i}\overset{1}{T}\alpha^{j}, \\ \overset{2}{S} = \sum_{j} \overset{2}{a}{}^{i}{}_{j}\alpha^{i}\overset{2}{T}\alpha^{j}, \end{cases} \qquad (i = 0, 1, 2, 3).$$

Now,

$$\text{(3.14)} \qquad \begin{cases} \overset{1}{S}\overset{2}{S} = \sum_{j} \overset{1}{a}{}^{i}{}_{j}\alpha^{i}\overset{1}{T}\alpha^{j}\overset{2}{S} \\ \quad = \overset{1}{a}{}^{i}{}_{0}\alpha^{i}\overset{1}{T}\alpha^{0}\left(\sum_{k} \overset{2}{a}{}^{0}{}_{k}\alpha^{0}\overset{2}{T}\alpha^{k}\right) \\ \quad + \overset{1}{a}{}^{i}{}_{1}\alpha^{i}\overset{1}{T}\alpha^{1}\left(\sum_{k} \overset{2}{a}{}^{1}{}_{k}\alpha^{1}\overset{2}{T}\alpha^{k}\right) \\ \quad + \overset{1}{a}{}^{i}{}_{2}\alpha^{i}\overset{1}{T}\alpha^{2}\left(\sum_{k} \overset{2}{a}{}^{2}{}_{k}\alpha^{2}\overset{2}{T}\alpha^{k}\right) \\ \quad + \overset{1}{a}{}^{i}{}_{3}\alpha^{i}\overset{1}{T}\alpha^{3}\left(\sum_{k} \overset{2}{a}{}^{3}{}_{k}\alpha^{3}\overset{2}{T}\alpha^{k}\right) \\ \quad = \sum_{k=0}^{3}\sum_{j=0}^{3} \overset{1}{a}{}^{i}{}_{j}\overset{2}{a}{}^{j}{}_{k}\alpha^{i}\overset{1}{T}\overset{2}{T}\alpha^{k}. \end{cases}$$

In (3.14) we have used the fact that $(\alpha^{i})^{2} = I$. If we denote the coefficients of the transformation L by $a^{i}{}_{j}$ we have from $L = \overset{1}{L}\overset{2}{L}$

$$\text{(3.15)} \qquad a^{i}{}_{j} = \sum_{k=0}^{3} \overset{1}{a}{}^{i}{}_{k}\overset{2}{a}{}^{k}{}_{j},$$

and hence

$$\text{(3.16)} \qquad \overset{1}{S}\overset{2}{S} = \sum_{j=0}^{3} a^{i}{}_{j}\alpha^{i}\overset{1}{T}\overset{2}{T}\alpha^{j}.$$

But equations (3.16) are just equations (3.6) for S and T for the transfor-

mation L which we see is satisfied by

$$\text{(3.17)} \qquad \begin{cases} S = \overset{1}{S}\,\overset{2}{S} \\ T = \overset{1}{T}\,\overset{2}{T} \end{cases}$$

for all i. Furthermore, since $\overset{1}{S}$ and $\overset{2}{S}$ are orthogonal and satisfy (3.6a), S will also be orthogonal and satisfy (3.6a). Hence we have proved our lemma.

Now using Lemmas 1 and 2 it is clear how one may prove that for proper Lorentz transformations equations (3.6) and (3.a) have the solutions given by the theorem. One can show by direct calculation that for rotations involving only two axes, equations (3.6) and (3.6a) have a solution for T given by (3.7) and the same solution for S is given by any of the equations (3.6). Furthermore, the matrix S is orthogonal (*not* unitary), if λ is taken to be unity. For an arbitrary proper Lorentz transformation, one uses lemma 1 and then 2 to obtain (3.7) for T and to show that equations (3.6) have as a solution for S an orthogonal matrix within a scalar factor λ.

To show that the matrices S form a representation of a group, when $\lambda = 1$, one must show

a) If the Lorentz transformation is the identity, S is the identity. One proves this statement by direct calculation using equations (3.6) and (3.6a).

b) If the Lorentz transformation L leads to S, then L^{-1} leads to the inverse matrix S^{-1}. This statement is proved in much the same way as Lemma 2.

c) A product of Lorentz transformations leads to a product of corresponding matrices S. Lemma 2 essentially proves this statement.

We have not yet shown that the solutions S and T of (3.6) and (3.6a) are unique within a factor λ and are given by (3.12). The uniqueness can be proved by considering the operators

$$\widetilde{S} = S\,\overset{n}{S}^{-1}\overset{n-1}{S}^{-1} \ldots \overset{1}{S}^{-1}, \qquad \widetilde{T} = T\,\overset{n}{T}^{-1}\overset{n-1}{T}^{-1} \ldots \overset{1}{T}^{-1}.$$

By methods similar to those used in the proof of Lemma 2, one can show

$$\text{(3.18)} \qquad \widetilde{S} = \alpha^i \widetilde{T} \alpha^i, \qquad (i = 0, 1, 2, 3).$$

Furthermore

$$\text{(3.18}a\text{)} \qquad \widetilde{S}_{01} = \widetilde{S}_{02} = \widetilde{S}_{03} = 0\,.$$

Then by direct calculation one can show from (3.18) and (3.18a) that

$$\tilde{T} = \lambda I = \tilde{S}\,, \tag{3.19}$$

where λ is an arbitrary scalar. Hence the uniqueness is proved.

Having obtained the transformation properties of ψ, we can now describe the spin of the electromagnetic field.

Let us consider an infinitesimal rotation $\delta\varphi$ around the x^i space axis. Then the component of angular momentum M^i corresponding to this axis is defined by ([9])

$$M^i\psi\delta\varphi = \frac{1}{i}\,\delta\psi\,, \tag{3.20}$$

where $\delta\psi$ is the change in ψ. In general one can write

$$M^i = \frac{1}{i}\,\frac{\partial}{\partial\varphi} + \sigma^i\,. \tag{3.21}$$

The matrix σ^i is defined to be the component of the spin about the axis x^i. In our particular case, we know that for any space-like rotation (using $\lambda = 1$)

$$S = T = (a^i{}_j)\,. \tag{3.22}$$

On considering small rotations about the space axes, and using (3.22), (3.21), (3.20), we obtain as the three spin components

$$\left\{\begin{array}{l} \sigma^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \\ \sigma^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{array}\right. \tag{3.23}$$

It is easily verified that

$$\boldsymbol{\sigma} \times \boldsymbol{\sigma} = i\boldsymbol{\sigma} \tag{3.24}$$

([9]) P. A. M. Dirac: loc. cit., Section 35.

and

$$(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 = 2 = l(l+1)\,, \qquad (l=1). \tag{3.25}$$

Furthermore, it is clear that each component of the spin has the eigenvalues ± 1, 0. Hence the electromagnetic field is a field of spin 1 as required.

Incidentally, the operator $\sigma \cdot P = (1/i) \sum_{j=1}^{3} \sigma^j (\partial/\partial x^j)$ commutes with the Hamiltonian H_0 as in the usual Dirac theory.

4. – Improper Lorentz transformations.

Up to now we have considered only proper Lorentz transformations. Since improper Lorentz transformations can be considered as a product of a proper transformation by an elementary improper one, we shall restrict our attention to the elementary improper transformations. We shall merely summarize the results.

There are four elementary improper Lorentz transformations. We shall consider them in order.

4·1. *Space and time reversal.* – In such a transformation we have

$$x^{i'} = -x^i\,, \qquad (i=0,\,1,\,2,\,3). \tag{4.1}$$

We again make the Ansatz

$$\begin{cases} \psi' = S\psi \\ \Phi' = T\psi\,. \end{cases} \tag{4.2}$$

The equations for S and T analogous to (3.6) and (3.6*a*) are

$$\begin{cases} S = \alpha^i T \alpha^i\,, & (i=0,\,1,\,2,\,3), \\ S_{01} = S_{02} = S_{03} = 0\,, \end{cases} \tag{4.3}$$

for which the solution is

$$T = \lambda I = -S\,, \tag{4.4}$$

where λ is an arbitrary constant. If we choose $\lambda = 1$, then Φ transforms like a four vector.

In the remaining improper transformations, the Ansatz (4.2) will not work,

for it leads to inconsistent equations for S and T. Instead we must assume an anti-linear relationship ψ' between ψ and we write

$$\text{(4.5)} \qquad \begin{cases} \psi' = S\psi^* \text{, (asterisk means complex conjugate),} \\ \Phi' = T\Phi \text{,} \\ \psi_0' = 0 \text{,} \end{cases}$$

for the remaining improper transformations which we now consider.

4·2. *Time reversal.* – Here we have

$$\text{(4.6)} \qquad \begin{cases} x^{0'} = -x^0 \\ x^{i'} = x^i \end{cases} \qquad (i = 1, 2, 3).$$

The equations for S and T are

$$\text{(4.7)} \qquad \begin{cases} S = T \\ S = -\alpha^i S \alpha^{i*} \text{,} \\ S_{01} = S_{02} = S_{03} = 0 \end{cases} \qquad (i = 1, 2, 3),$$

for which the solution is

$$\text{(4.8)} \qquad S = T = \lambda \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we pick $\lambda = 1$, then the sources transform like a four-vector, the electric field E reverses sign, while the magnetic field H remains unchanged under the transformation.

4·3. *Space inversion.* – In this case

$$\text{(4.9)} \qquad \begin{cases} x^{0'} = x^0 \text{,} \\ x^{i'} = -x^i \text{,} \end{cases} \qquad (i = 1, 2, 3),$$

and on using (4.5) our equations for S and T are

$$\begin{cases} S = -T\,, \\ S = -\alpha^i S \alpha^{i*}\,, \\ S_{01} = S_{02} = S_{03} = 0\,, \end{cases} \qquad (i = 1, 2, 3),$$

for which the solution is

$$S = -T = \lambda \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.11}$$

On picking $\lambda = 1$, Φ transforms like a four-vector, E transforms like a vector, while H transforms like a pseudo-vector.

4·4. *Space reflections.* – In this case we invert a single space axis, the x^1 axis, for example, while the other axes remain unchanged.

The equations for S and T are

$$\begin{cases} S = -T \\ S = \alpha^2 S \alpha^{2*} \\ S = \alpha^3 S \alpha^{3*} \\ S = -\alpha' S \alpha'^{*} \\ S_{01} = S_{02} = S_{03} = 0 \end{cases} \tag{4.12}$$

for which the solutions are

$$S = -T = \lambda \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.13}$$

If $\lambda = 1$, then Φ transforms like a four-vector, $\boldsymbol{E}$ again transforms as a vector, and $\boldsymbol{H}$ as a pseudo-vector.

5. – Vector potentials.

It is quite easy to introduce the vector potential in terms of the spinor notation. We shall summarize some results. Let us denote, as usual, the

electromagnetic field by ψ where

$$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \qquad \psi_0 = 0 \,. \tag{5.1}$$

Then the (generally complex) vector potential A

$$A = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} \tag{5.2}$$

is defined to be that vector which satisfies

$$\psi' = \frac{1}{i} \sum_{j=0}^{3} \alpha^j \frac{\partial}{\partial x_j} A \,, \tag{5.3}$$

where ψ' is the same as ψ except for the top component:

$$\psi' = \begin{pmatrix} \psi_0' \\ \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix}, \tag{5.4}$$

$$\begin{cases} \psi_i' = \psi_i \,, & (i = 1, 2, 3), \\ \psi_0' \quad \text{an arbitrary function.} \end{cases} \tag{5.4a}$$

If A is real, then one can show that the field strengths as given by (5.3) and (5.4) are related to the four vector A in the usual way, i.e.

$$F^{ij} = \frac{\partial \varphi^j}{\partial x_i} - \frac{\partial \varphi^i}{\partial x_j} \,, \tag{5.5}$$

where

$$\varphi^i = A_i \,, \tag{5.6}$$

$$F^{ij} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix}, \qquad (i = 0, 1, 2, 3). \tag{5.6a}$$

If, on the other hand A is purely imaginary,

$$A = iB\,, \qquad B \text{ real:} \tag{5.7}$$

then the components of B are the components of the anti-potential which one can introduce if there are no external sources. We shall discuss the anti-potential in more detail shortly.

The most general gauge transformations correspond to changes in A which leave the last three components of ψ' unaltered, but which may in general alter the arbitrary function ψ'_0.

An example of a gauge transformation which will not alter *any* component of ψ' is given by

$$A' = A - \frac{1}{i}\sum_{j=0}^{3} \alpha^j \frac{\partial}{\partial x^j} \Lambda\,, \tag{5.8}$$

where A' is the new potential and where Λ is a four component entity which obeys the wave equation:

$$\Box^2 \Lambda = 0\,. \tag{5.9}$$

This statement is proved on using A' in (5.3) and then using (2.9). This gauge transformation is the one most frequently used.

The vector potential A can be solved for in terms of ψ' by using any Green's function $G(\boldsymbol{x}, t\,|\,\boldsymbol{x}', t')$ which satisfies

$$-\frac{1}{i}\sum_{j=0}^{3} \alpha^j \frac{\partial}{\partial x_j} G(\boldsymbol{x}, t\,|\,\boldsymbol{x}', t') = \delta(\boldsymbol{x} - \boldsymbol{x}')\,\delta(t - t')\,, \tag{5.10}$$

for then we may write

$$A(\boldsymbol{x}, t) = \iint G(\boldsymbol{x}, t\,|\,\boldsymbol{x}', t')\ \psi'(\boldsymbol{x}', t')\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}t\,. \tag{5.11}$$

In (5.11) and elsewhere $\boldsymbol{x}$ refers to the space variables only.

That an infinity of Green's functions satisfying (5.10) exist is proved in the usual treatments of Dirac's equations. Each Green's function will lead to a different vector potential A, all of which, however, are equivalent in that

they lead to the same field strengths and, in fact, even to the same arbitrary function ψ_0'. The corresponding gauge transformation is usually more general than (5.8).

If one requires the vector potential A to satisfy the Lorentz condition

$$\frac{\partial A_0}{\partial t} + \operatorname{div} \boldsymbol{A} = \sum_{i=0}^{3} \frac{\partial A_i}{\partial x^i} = 0 \,, \tag{5.12}$$

then $\psi_0' = 0$, $\psi' \equiv \psi$ and

$$\psi = -\frac{1}{i} \sum_{j=0}^{3} \alpha^j \frac{\partial}{\partial x_j} A \,. \tag{5.13}$$

On multiplying both sides of (5.13) by $(-1/i) \sum_{j=0}^{3} \alpha^j (\partial/\partial x^j)$, using Maxwell's equations (2.2), and (2.9), we obtain

$$\Box^2 A = -4\pi \Phi \,. \tag{5.14}$$

These are the usual equations for the vector potential which satisfy the Lorentz condition. Our potential, however, may be complex.

5·1. *Relation between the potential and anti-potential.* – If there are no sources, one can introduce instead of the usual electromagnetic tensor F^{ij} and potential φ^i (equations (5.5), (5.6), (5.6a)), another electromagnetic tensor $\hat{F}^{ij}$ where the electric and magnetic fields are interchanged with a suitable change in sign,

$$\hat{F}^{ij} = \begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & E_z & -E_y \\ H_y & -E_z & 0 & E_x \\ H_z & E_y & -E_x & 0 \end{pmatrix}, \qquad (i, j = 0, 1, 2, 3). \tag{5.15}$$

The real anti-potential ν^i is defined by

$$\hat{F}^{ij} = \frac{\partial \nu^j}{\partial x_i} - \frac{\partial \nu^i}{\partial x_j} \,. \tag{5.16}$$

We can now give a relation between the potential and the anti-potential which gives rise to the same electromagnetic field. We shall assume both potentials to satisfy the Lorentz condition. Denoting by φ a 4-component real column vector whose components are the components of the potential, and

by ν the real column vector whose components are components of the anti-potential, the relation is

$$\varphi(\boldsymbol{x}, t) = -\frac{1}{i}\frac{\partial}{\partial t}\sum_{j=1}^{3}\sigma^j\frac{\partial}{\partial x^j}\int\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\nu(\boldsymbol{x}', t)\,\mathrm{d}\boldsymbol{x}', \tag{5.17}$$

where σ^j are the spin matrices (3.23) and $\boldsymbol{x}$, $\boldsymbol{x}'$ refer to space co-ordinates only.

5·2. *A real vector potential corresponding to an electromagnetic field.* – We shall give a real vector potential from which an electromagnetic field can be derived in the manner of (5.3) or equivalently (5.5). Let ψ be the spinor which gives the electromagnetic field in the sense (2.1), (2.1a). Then a real vector potential $A(=\varphi)$, which will reproduce this field is

$$A(\boldsymbol{x}, t) = \mathrm{Re}\left\{\frac{1}{i}\sum_{j=1}^{3}\alpha^j\frac{\partial}{\partial x^j}\int\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\psi(\boldsymbol{x}', t)\,\mathrm{d}\boldsymbol{x}'\right\}, \tag{5.18}$$

where Re means real part and α^j are the matrices given in (2.3). The vector potential given in (5.18) does not satisfy the Lorentz condition. Instead its gauge is such that the 0-th component is the electrostatic potential, i.e.,

$$\begin{cases} A_0(\boldsymbol{x}, t) \quad \varphi^0 \qquad -\int\dfrac{\varrho(\boldsymbol{x}', t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}\boldsymbol{x}'\,, \\ \operatorname{div}\boldsymbol{A} = 0\,, \qquad \boldsymbol{A} = (A_1, A_2, A_3)\,. \end{cases} \tag{5.19}$$

* * *

The writer wishes to express his thanks to Professors K. O. Friedrichs and W. Magnus and Dr. R. Kraichnan of New York University, and Professor M. Marcuvitz of Brooklyn Polytechnic Institute for helpful discussions.

Appendix

Elements of matrix S for a proper Lorentz transformation ($\lambda = 1$).

$$S_{00} = 1\,.$$

$$S_{01} = S_{02} = S_{03} = S_{10} = S_{20} = S_{30} = 0\,.$$

$$S_{11} = (a^1{}_1 a^0{}_0 - a^1{}_0 a^0{}_1) + i(a^1{}_2 a^0{}_3 - a^1{}_3 a^0{}_2) .$$

$$S_{12} = (a^1{}_2 a^0{}_0 - a^1{}_0 a^0{}_2) + i(a^1{}_3 a^0{}_1 - a^1{}_1 a^0{}_3) .$$

$$S_{13} = (a^1{}_3 a^0{}_0 - a^1{}_0 a^0{}_3) + i(a^1{}_1 a^0{}_2 - a^1{}_2 a^0{}_1) .$$

$$S_{21} = (a^2{}_1 a^0{}_0 - a^2{}_0 a^0{}_1) + i(a^2{}_2 a^0{}_3 - a^2{}_3 a^0{}_3) .$$

$$S_{22} = (a^2{}_2 a^0{}_0 - a^2{}_0 a^0{}_2) + i(a^2{}_3 a^0{}_1 - a^2{}_1 a^0{}_3) .$$

$$S_{23} = (a^2{}_3 a^0{}_0 - a^2{}_0 a^0{}_3) + i(a^2{}_1 a^0{}_2 - a^2{}_2 a^0{}_1) .$$

$$S_{31} = (a^3{}_1 a^0{}_0 - a^3{}_0 a^0{}_1) + i(a^3{}_2 a^0{}_3 - a^3{}_3 a^0{}_2) .$$

$$S_{32} = (a^3{}_2 a^0{}_0 - a^3{}_0 a^0{}_2) + i(a^3{}_3 a^0{}_1 - a^3{}_1 a^0{}_3) .$$

$$S_{33} = (a^3{}_3 a^0{}_0 - a^3{}_0 a^0{}_3) + i(a^3{}_1 a^0{}_2 - a^3{}_2 a^0{}_3) .$$

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Statistical Weight of a System of Particles with Arbitrary Spins.

V. S. Barašenkov and B. M. Barbašev

Joint Institute for Nuclear Research
Laboratory of Theoretical Physics - Dubna, USSR

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In some cases it is important to know the number of possible states of a system of particles with a given angular or isotopic spin. An instance of this is the multiple production theory ([1,2]) the analysis on the basis of the charges of separate particles ([3,4]) being often impossible because of considerable complexity (*). The statistical weights of the systems of particles of spin $S = \frac{1}{2}$; $S = 1$ have been studied in details in papers ([5]) and ([6]).

The formulae which have been obtained in these works are complicated and the drawing up of tables of statistical weights for different numbers of particles proves rather tedious.

The difficulties considerably increase if the particles of spin $S = \frac{3}{2}$ are also taken into account, which appears to be indispensable to bring the mul- tiple production theory in agreement with experiment ([3,4]). However,

([1]) E. Fermi: *Progr. Theor. Phys.*, **5**, 570 (1951). S. Z. Belenkij and L. D. Landau: *Usp. Fiz. Nauk*, **56**, 309 (1955).

([2]) S. Z. Belenkij and A. L. Nikišov: *Žu. Éksper. Teor. Fiz.*, **28**, 744 (1955).

([3]) V. M. Maksimenko and A. I. Nikišov: *Žu. Éksper. Teor. Fiz.*, **31**, 727 (1956).

([4]) V. S. Barašenkov, B. M. Barbašev, E. G. Bubelev and V. M. Maksimenko: to be published in *Žu. Éksper Teor. Fiz.*

(*) Sometimes a partial analysis of the system charge states turns out to be possible by separating a part of the particles into groups (e.g. pions and nucleons from strange particles) which are characterized by their total isotopic spin. No charge analysis is carried out within each group. The statistical weight of such a group is determined by the number and kind of its particles and the value of the total isotopic spin. The isotopic spin of the group may be considered.

([5]) G. Racah: *Rend. Lincei*, **17**, 386 (1933).

([6]) Y. Yeivin and A. de Shalit: *Nuovo Cimento*, **1**, 1147 (1955).

$m = 1$

S	0			$\frac{1}{2}$			1			$\frac{3}{2}$			
n \ v	1	3	5	0	2	4	1	3	5	0	2	4	1
0	—	1	4	—	1	4	1	3	10	1	2	6	1
1	1	3	10	1	3	10	2	7	24	1	4	14	2
2	2	7	24	2	7	24	5	17	59	3	10	35	5
3	5	17	59	5	17	59	12	42	148	7	25	89	13
4	12	42	148	12	42	148	30	106	378	18	64	230	34
5	30	106	378	30	106	378	76	272	980	46	166	602	90
6	76	272	980	76	272	980	196	708	2 573	120	436	2 593	240
7	196	708	2 573	196	708	2 573	512	1 865	6 828	316	1 157	4 255	645
8	512	1 865	6 828	512	1 865	6 828	1 353	4 963	18 286	841	3 098	11 458	1 745
9	1 353	4 963	18 286	1 353	4 963	18 286	3 610	13 323	49 360	2 257	8 360	31 074	4 750
10	3 610	13 323	49 360	3 610	13 323	49 360	9 713	36 037	134 160	6 103	22 714	84 800	13 001

$m = 2$

S	0			$\frac{1}{2}$			1			$\frac{3}{2}$			
n \ v	0	2	4	1	3	5	0	2	4	1	3	5	0
0	1	2	6	2	6	20	1	4	14	2	8	29	1
1	1	4	14	4	14	49	3	10	35	6	21	75	3
2	3	10	35	10	35	124	7	25	89	15	54	195	8
3	7	25	89	25	89	319	18	64	230	39	141	513	21
4	18	64	230	64	230	832	46	166	602	102	372	1 363	56
5	46	166	602	166	602	2 195	120	436	1 593	270	991	3 653	150
6	120	436	2 593	436	1 593	5 848	316	1 157	5 255	721	2 662	9 865	405
7	316	1 157	4 255	1 157	4 255	15 713	841	3 098	11 458	1 941	7 203	26 819	1 100
8	841	3 098	11 458	3 098	11 458	42 532	2 257	8 360	31 074	5 262	19 616	73 342	3 005
9	2 257	8 360	31 074	8 360	31 074	115 874	6 103	22 714	84 800	14 354	53 726	201 626	8 251
10	6 103	22 714	84 800	22 714	84 800	317 500	16 611	62 086	232 700	39 372	147 900	556 910	22 761

2		$\frac{5}{2}$			3			$\frac{7}{2}$			4		
3	5	0	2	4	1	3	5	0	2	4	1	3	5
3	10	0	1	4	0	1	5	0	0	1	0	0	1
7	25	1	3	11	1	4	16	0	1	5	0	1	6
18	65	2	8	30	3	12	47	1	4	17	1	5	23
47	171	6	22	82	9	35	135	3	13	53	4	18	77
124	454	16	60	224	26	100	383	10	40	159	14	59	243
330	1 215	44	164	613	74	283	1 080	30	119	467	45	184	742
885	3 275	120	449	1 682	209	797	3 037	89	348	1 355	139	558	2 221
2 390	8 885	329	1 233	4 630	588	2 240	8 533	259	1 007	3 903	419	1 663	6 565
6 495	24 246	904	3 397	12 788	1 652	6 293	23 982	748	2 896	11 194	1 244	4 901	19 238
17 751	66 514	2 493	9 391	35 440	4 641	17 689	67 466	2 148	8 298	32 026	3 657	14 337	56 070
48 763	183 340	698	8 26 049	98 540	138 04	49 777	190 050	6 150	23 728	91 501	10 680	41 733	162 810

2		$\frac{5}{2}$			3			$\frac{7}{2}$			4		
2	4	1	3	5	0	2	4	1	3	5	0	2	4
4	15	2	7	26	1	3	11	1	4	16	0	1	5
11	40	5	19	71	2	8	31	3	12	48	1	4	17
29	106	14	52	194	6	23	88	9	36	142	3	13	54
77	283	38	142	531	17	65	248	27	106	414	10	41	166
206	761	104	389	1 458	48	183	697	79	308	1 196	31	125	499
555	2 060	285	1 069	4 017	135	514	1 957	229	888	3 436	94	374	1 479
1 505	6 610	784	2 948	11 106	379	1 443	6 496	659	2 548	9 839	280	1 105	4 343
4 105	15 361	2 164	8 158	30 810	1 064	4 053	15 449	1 889	7 291	28 123	825	3 238	12 674
11 256	42 268	5 994	22 652	85 752	2 989	11 396	43 484	5 402	20 832	80 316	2 413	9 436	36 832
31 012	116 826	16 658	63 100	239 410	8 407	32 088	122 584	15 430	59 484	229 324	7 023	27 396	106 740
85 814	324 210	46 442	176 310	670 360	23 681	90 496	346 150	44 054	169 840	654 922	20 373	79 344	308 772

$m = 3$

S	0			$\frac{1}{2}$			1			$\frac{3}{2}$			
n \ v	1	3	5	0	2	4	1	3	5	0	2	4	1
0	2	8	29	2	8	29	6	21	75	4	13	46	7
1	6	21	75	6	21	75	15	54	195	9	33	120	18
2	15	54	195	15	54	195	39	141	513	24	87	318	48
3	39	141	513	39	141	513	102	372	1363	63	231	850	129
4	102	372	1363	102	372	1363	270	991	3653	168	619	2290	349
5	270	991	3653	270	991	3653	721	2662	9865	451	1671	6212	950
6	721	2662	9865	721	2662	11865	1941	7203	26819	1220	4541	20954	2600
7	1941	7203	26819	1941	7203	26819	5262	19616	73342	3321	12413	46523	7151
8	5262	19616	73342	5262	19616	73342	14354	53726	201626	9092	34110	128284	19756
9	14354	53726	201626	14354	53726	201626	39372	147900	556910	25018	94174	355284	54802
10	39372	147900	556910	39372	147900	556901	108528	409010	1544770	61569	261110	987860	152582

$m = 4$

S	0			$\frac{1}{2}$			1			$\frac{3}{2}$			
n \ v	0	2	4	1	3	5	0	2	4	1	3	5	0
0	4	13	46	13	46	166	9	33	120	20	74	272	1
1	9	33	120	33	120	438	24	87	318	54	198	730	3
2	24	87	318	87	318	1168	63	231	850	144	532	1972	8
3	63	231	850	231	850	3140	168	619	2290	388	1440	5362	22
4	168	619	2290	619	2290	8502	451	1671	6212	1052	3922	14664	60
5	451	1671	6212	1671	6212	23166	1220	4541	16954	2870	10742	40311	165
6	1220	4541	20954	4541	16954	63477	3321	12413	55523	7872	29569	111330	455
7	3321	12413	46523	12413	46523	174807	9092	34110	128284	21697	81761	308761	1260
8	9092	34110	128284	34110	128284	483568	25018	94174	355284	60064	227000	859576	3504
9	25018	94174	355284	94174	355284	1343144	69156	261110	987860	166936	632576	2401318	9778
10	69156	261110	987860	261110	987860	3744462	191954	726750	2756602	465640	1768742	6729596	27368

2			$\frac{5}{2}$			3			$\frac{7}{2}$			4		
3	5	0	2	4	1	3	5	0	2	4	1	3		
25	91	3	12	45	5	20	77	2	8	32	3	12	49	
66	243	9	33	123	15	57	217	6	24	94	9	37	149	
177	655	24	90	337	42	160	609	18	70	272	28	112	448	
478	1777	66	247	927	118	449	1709	52	202	782	84	333	1313	
1299	4849	181	680	2559	331	1260	4799	150	580	2240	249	980	3844	
3550	13301	499	1879	7089	929	3539	13492	430	1660	6403	731	2864	11195	
9751	36658	1380	5210	22704	2610	9[illegible]3	37988	1230	4743	18284	2133	8331	32488	
26907	101465	3830	14494	54942	7343	28035	107135	3513	13541	52193	6198	24158	94066	
74558	281942	10664	40448	153658	20692	79100	302666	10028	38652	149008	17960	69908	271944	
207384	786234	29784	113210	430950	58408	223566	856548	28624	110356	425598	51948	202032	785448	
578850	2199692	83426	317740	2111832	165158	632982	2428224	81732	315242	1216392	150084	583410	2267460	

2			$\frac{5}{2}$			3			$\frac{7}{2}$			4		
2	4	1	3	5	0	2	4	1	3	5	0	2		
41	152	21	78	292	10	37	140	16	62	240	6	25	10	
111	412	57	214	804	27	103	392	46	178	688	19	75	29	
301	1122	157	590	2222	76	289	1100	132	510	1968	56	221	86	
821	3072	433	1632	6162	213	811	3090	378	1458	5621	165	647	253	
2251	8452	1199	4530	17145	598	2279	8693	1080	4163	16044	482	1884	735	
6201	23357	3331	12615	47853	1681	6414	24496	3083	11881	45790	1402	5467	2129	
17156	73807	9284	35238	133954	4733	18082	69147	8798	33909	130724	4065	12827	6157	
47651	180477	25954	98716	376008	13349	51065	195531	25111	96815	373405	11762	45750	17787	
132826	504292	72762	277292	1058174	37716	144466	553882	71704	276590	1067384	33988	132124	51350	
371466	1413458	204530	780882	2985134	106750	409416	1571676	204886	790794	3053700	98136	381378	1482 02	
041992	3972994	576332	2204252	8440152	302666	1162260	4467158	585908	2262906	8744360	283242	1100646	4277 20	

the drawing up of tables of statistical weights for systems of particles with different spins can be substantially simplified by using recurrent relations.

The statistical weight $W_{\xi_1 \dots \xi_k \dots \xi_n}(S)$ of the system $(\xi_1 \dots \xi_k \dots \xi_n)$ that consists of ξ_1 particles of spin S_1, ξ_2 particles of spin S_2, etc., for the given total spin of the whole system may be defined as a sum of statistical weights $W_{\xi_1 \dots \xi_k - 1 \dots \xi_n}(S_i)$ of the system $(\xi_1 \dots \xi_k - 1 \dots \xi_n)_{S_i}$ from which we may pass to one of the states of the system $(\xi_1 \dots \xi_k \dots \xi_n)_S$ by adding a particle of spin S_k:

$$W_{\xi_1 \dots \xi_k \dots \xi_n}(S) = \sum_{I_k} W_{\xi_1 \dots \xi_k - 1 \dots \xi_n}(S_i) \,. \tag{1}$$

It follows from the definition of the spin addition law that $S_i = |S - S_k|; |S - (S_k - 1)| \dots (S + S_k)$. Evidently $W_{\xi_1 \dots \xi_k - 1 \dots \xi_n}(S_i) = 0$ if the system $(\xi_1 \dots \xi_k \dots \xi_n)_S$ cannot have the spin value $S = S_i$.

Further, let us assume $W_{0.0 \dots 0}(0) = 1$. In a specific case when $(\xi_1 \dots \xi \dots \xi_n) = (\xi_{S=\frac{1}{2}}; 0) = (\nu, 0)_S$ and $(\xi_1; \dots \xi_n)_S = (0, \xi_{S=1}) = (0, n)_S$, relation (1) is identical to the corresponding relations in [6].

Below we give the tables of values $W_{\xi_{S=\frac{1}{2}} \xi_{S=1} \xi_{S=\frac{3}{2}}}(S) = W_{\nu,n,m}(S)$ which have been obtained with the help of the recurrent relation (1) for the values of ν, m, n and which are necessary to compute the multiple production of particles at energies $E \lesssim 10 \div 15$ GeV.

The tables do not contain the values of $W_{\nu.n.0}(S)$. These are presented in [6] (*). The calculations have shown that the use of the recurrent relation (1) is more convenient than that of closed formulae for $W_{\xi_1 \dots \xi_n}(S)$ (e.g. of formulae (1), (7) and (9) in [6]) and considerably simplifies the drawing up of tables of statistical weights [1]. In a number of cases the calculations for particles of large spin are simplified by preliminary transformation of the statistical weight expressions on the right-hand side of relation (1) with the help of the recurrent relation (1).

* * *

In conclusion the authors take pleasure in thanking Prof. BLOHINCEV for his interest in the work.

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(*) The tables of the work [3] were computed anew with the help of recurrent relations and the necessity was found to make corrections:

$W_{2,2}(2) = 3$; $W_{4,4}(5) = 7$; $W_{2,10}(1) = 5\,798$; $W_{3,10}(\frac{3}{2}) = 49\,360$; $W_{7,10} = 21\,726$.

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On the Theory of Stochastic Processes in Nuclear Reactors.

L. PÁL

Central Research Institute of Physics
of the Hungarian Academy of Sciences - Budapest

(ricevuto il 18 Aprile 1957)

1. – The classical part of the theory of nuclear reactors (*) is founded on the Boltzmann-equation, well-known from the kinetic theory of gases. Essentially all practical problems can be reduced to the search for approximative solutions of that equation fulfilling various initial and boundary conditions.

A considerable number of papers (1) have been publised on the various methods of approximation and actual calculations performed with these methods. A few monographs (2) have also appeared which summarize systematically the results so far published. However, up to the present the processes occurring in nuclear reactors have not been considered in terms of the probability theory. Certain initial attempts were made in this field as well (3-5) no

(*) The non-classical part of the theory of nuclear reactors, namely, the theory of cross-sections etc., belongs to the field of theoretical nuclear physics.

(1) R. E. MARSHAK: *Rev. Mod. Phys.*, **19**, 185 (1947); A. M. WEINBERG and L. C. NODERER: *Theory of Neutron Chain Reactions*, AECD-3471 (1951); O. OLSSON: *Ark. för Fys.*, **10**, 129 (1955).

(2) S. GLASSTONE and M. E. EDLUND: *The Elements of Nuclear Reactor Theory*, (New York, 1952); H. SOODAK and E. C. CAMPBELL: *Elementary Pile Theory* (New York, 1950); *Selected Reference Material United States Atomic Energy Program*, Vol. 2. *Reactor Handbook*: *Physics* (1955).

(3) V. V. ČAVTANIDZE and O. D. ĆEJŠVILI: *Žu. Èxper. Teor. Fiz.* **28**, 3 (1955).

(4) L. PÁL: *Publiactions de l'Institut Mathématique de l'Academie des Sciences de Hongrie*, **1**, 41 (1956).

(5) L. TAKÁCS: *Publications de l'Institut Mathématique de l'Academie des Science de Hongrie*, **1**, 55 (1956).

results, however, have been obtained, except when the simplest processes are involved.

One of the most general formulations of Boltzmann's transport equation describing the variation in time and space of the average neutron density was given by L. N. USSACHOFF (6) who generalized the transport equation in such a way that at the same time he considered the fact of neutron multiplication and the role of delayed neutrons as well. Obviously the solution of this generalized transport equation can be given so much the more only under certain strongly simplifying assumptions.

In the course of developing solution methods for the transport equations numerous authors (7,8), have made use of the adjoint of the transport equation. According to USSACHOFF, it was B. B. KADOMTSEV who for the first time (in 1955) pointed out the physical significance of the adjoint equation. USSACHOFF's paper quoted above gave an extremely clear formulation of the physical contents of the adjoint transport equation. He introduced the concept of « neutron importance » and proved that the adjoint transport equation is equivalent to the integro-differential equation for the neutron importance. The definition of neutron importance may be given in the most simple way as follows. Suppose that at a time $t = 0$ the power of the reactor is zero, i.e. no neutrons are present in the reactor. Now, if a neutron of energy E_0 and direction of velocity $\boldsymbol{\Omega}_0$ at a time $t = 0$ is injected in the volume element $\mathrm{d}V\boldsymbol{r}_0$ at $\boldsymbol{r}_0$, the importance of the neutron at a time t is determined by the average number of neutrons produced by this neutron.

We note that the definition of neutron importance used here somewhat differs from that given by USSACHOFF (6). It is worth mentioning that both the equations for neutron density and those for neutron importance are in close relation with certain generalized forms (10) of the Kolmogoroff-Feller equations (9) well-known from the theory of stochastic processes.

The purpose of this paper is to formulate in the most general manner the fundamental equations describing the stochastic processes in reactors, and to derive from these the generalized equations for the average neutron density and neutron importance, as well as for the fluctuation of neutron density. It would prove especially difficult to derive the equation describing the fluc-

(6) L. N. USSACHOFF: *Proceedings of the International Conference on the Peaceful Uses of Atomic Energy*, Vol. **5**, p. 503.

(7) A. M. WEINBERG: *Am. Journ. Phys.*, **20**, 401 (1952); R. EHRLICH and H. HURWITZ: *Nucleonics*, **12**, No. 2, 23 (1954).

(8) E. D. PENDLEBURY: *Proc. Phys. Soc.*, A **68**, 474 (1955).

(9) A. KOLMOGOROFF: *Math. Ann.*, **104**, 415 (1931); W. FELLER: *Math. Ann.*, **113**, 113 (1936).

(10) L. PÁL: *Žu. Éksper. Teor. Fiz.*, **30**, 362, (1956); *Soviet Phys.*, **3**, 222, (1956).

tuation of neutron density without the results of the general theory of stochastic processes in reactors.

2. – First of all, let us start from the assumption that the medium in which neutrons move is a mixture of r types of atomic nuclei. Denote by N_1, N_2, ..., N_r the average number of the different types of nuclei per unit volume. If the spatial distribution of the various nuclei in the medium is not homogeneous, the density of the nuclei in a point defined by the vector $\boldsymbol{\rho}_0$ should be denoted respectively by $N_1(\boldsymbol{\rho}_0)$, $N_2(\boldsymbol{\rho}_0)$, ..., $N_r(\boldsymbol{\rho}_0)$. Denote further by σ_1^s, σ_2^s, ..., σ_r^s and σ_1^a, σ_2^a, ..., σ_r^a the microscopic cross-section for the scattering and absorption of the different types of nuclei. Let $w_s^j(E_0, \boldsymbol{\Omega}_0; E, \boldsymbol{\Omega})\,\mathrm{d}E\,\mathrm{d}\Omega$ be the probability that the energy of a neutron which suffered elastic or inelastic collisions at the j-th nucleus, is inside the interval $(E, E+\mathrm{d}E)$ and the direction of its velocity is inside the elementary cone $\mathrm{d}\Omega$ about $\boldsymbol{\Omega}$ provided that before the collision the energy of the neutron was E_0 and its direction of velocity $\boldsymbol{\Omega}_0$.

It is expedient to introduce the following density function:

$$w_s(\boldsymbol{\rho}_0, E_0, \boldsymbol{\Omega}_0; E, \boldsymbol{\Omega}) = \frac{\sum_{j=1}^{r} N_j \sigma_j^s w_s^j(E_0, \boldsymbol{\Omega}_0; E, \boldsymbol{\Omega})}{\sum_{j=1}^{r} N_j \sigma_j^s}. \tag{2.1}$$

For the sake of simplicity we introduce for the notation of the microscopic cross-sections the quantities

$$Q_s = \sum_{j=1}^{r} N_j \sigma_j^s, \qquad Q_a = \sum_{j=1}^{r} N_j \sigma_j^a. \tag{2.2}$$

Although for the general treatment several types of fissionable nuclei ought to be taken into account, in practice, however, usually only one single type of fission material is present. Thus in our treatment we consider only one type of fissionable nucleus. Denote by $N_f = N_f(\boldsymbol{\rho}_0)$ (*) the density of nuclei of the fissionable substance at the point defined by the vector $\boldsymbol{\rho}_0$, and by $Q_f = N_f \sigma^f$ the macroscopic fission cross-section. The total cross-section is evidently

$$Q_0 = Q_s + Q_a + Q_f. \tag{2.3}$$

In general the quantity Q_0 depends on the energy of the neutron in a compli-

(*) $N_f(\boldsymbol{\rho}_0)$ is obviously one of the densities $N_1(\boldsymbol{\rho}_0), \ldots, N_r(\boldsymbol{\rho}_0)$.

cated manner, and in case the spatial distribution of nuclei is inhomogeneous it depends on the space co-ordinates as well.

Regarding the number, the energy distribution, etc., of neutrons arising in the fission process most general assumptions will be made. Be $p(E_0, k_0, ..., k_l)$ the probability that in a fission caused by a neutron of energy E_0, k_0 prompt neutrons and in succession k_1, k_2, ..., k_l fission products in group l with decay constants λ_1, λ_2, ..., λ_l will be produced. The delayed neutrons belonging to the i-th group are emitted by the fission products of the groups i. Evidently we have

$$\sum_{k_1, \ldots, k_l} p(E_0, k_0, ..., k_l) = 1 \tag{2.4}$$

and

$$\sum_{k_0, \ldots, k_l} (k_0 + ... + k_l) p(E_0, k_0, ..., k_l) = \sum_{i=0}^{l} \langle k_i \rangle , \tag{2.5}$$

where $\langle k_0 \rangle = r_0$ is the average number of prompt neutrons per one fission process, and $\langle k_i \rangle$ the average number of fission products emitting delayed neutrons of the i-th group, also related to one act of fission. It can be seen directly that

$$\langle k_i \rangle = \nu_i \qquad (i = 1, 2, ..., l), \tag{2.6}$$

ν_i being the average number of delayed neutrons of the i-th group. Denote by μ_i $(i = 0, 1, ..., l)$ the mean square of $\langle k_i^1 \rangle$ $(i = 0, 1, ..., l)$.

Suppose that both the prompt neutrons and the delayed neutrons are independent of each other. Let us denote by $w_f^0(E_0, \mathbf{\Omega}_0; E, \mathbf{\Omega}) \mathrm{d}E \mathrm{d}\Omega$ the probability that the energy of some of the prompt neutrons lies inside the energy interval $(E, E+\mathrm{d}E)$, and its direction of velocity inside the elementary cone $\mathrm{d}\Omega$ about the vector $\mathbf{\Omega}$ provided the energy of the initial neutron was E_0, its direction of velocity $\mathbf{\Omega}_0$.

The delayed neutrons belonging to the i-th group are emitted by fission products of type i. We denote by $w_f^i(E_0, \mathbf{\Omega}_0; E, \mathbf{\Omega}) \mathrm{d}E \mathrm{d}\Omega$ the probability that the energy of the delayed neutron emitted by the fission product of type i is inside the energy interval $(E, E+\mathrm{d}E)$, its direction of velocity inside the elementary cone $\mathrm{d}\Omega$ about the vector $\mathbf{\Omega}$ again under the assumption that the prime neutron had the energy E_0 and direction of velocity $\mathbf{\Omega}_0$. The question of a suitable choice for the actual form of the probability density functions $w_f^i(E_0, \mathbf{\Omega}_0; E, \mathbf{\Omega})$ $(i = 0,1, ..., l)$ will be dealt with when discussing the applications.

3. – There exists in many respects a considerable resemblance between stochastic processes taking place in reactors and the cascade processes of cosmic radiation. Thus it seems to be practicable to make use of the relevant results

of cosmic ray cascade processes as well in the theory of stochastic processes occurring in reactors.

Suppose that the space and energy distribution, as well as the distribution of the directions of velocity of the neutrons are known at a time $t = 0$. Since each neutron initiates independently of the others a chain reaction with a certain probability, it is sufficient to consider a process initiated by one single neutron only. Denote the energy of this neutron by E_0, the direction of its velocity by Ω_0, and let it be located at the time $t = 0$ in a point defined by the vector r_0. For the treatment of this chain reaction initiated by one neutron the cascade theory of Jánossy [11] is extremely well suited. This theory makes use of that characteristic of certain stochastic processes (Markoff-processes) that by a suitable choice of probability variables the process can be determined by means of the first discontinuity of these variables (the so-called first collision method).

This procedure is in fact not a new one since among the fundamental equations of the theory of stochastic processes the so-called first Kolmogoroff-Feller equation concerns exactly the first sudden change of probability variables. In case of processes which do not involve a change in the number of particles, the advantage of this method is not so evident; this advantage, however, is almost invaluable in the theory of the so-called cascade processes implying a change in the number of particles. It was JÁNOSSY [11] who has called attention to this circumstance for the first time.

In the following we also make use essentially of the results of Jánossy's cascade theory [11]. However, our method slightly differs from that developed by JÁNOSSY. One of the differences is that, while JÁNOSSY chooses some of the space co-ordinates (approximately the direction parallel to the axis of the cascade) as parameter for describing the process, we choose, of course, the time as parameter. The application of one of the space co-ordinates as a parameter involves also in the cascade theory certain difficulties, as during the development of the process not only an increase but a decrease of the space co-ordinate may occur, although in cosmic ray cascades the probability of a backward scattering of particles along the axis of the cascade is insignificant. Thus the possible backward scattering may, as « catastrophic collisions », be excluded from the process.

However, in the theory of the slowing down and diffusion, as well as of the multiplication of neutrons this method of exclusion can by no means be utilized since, unlike cosmic ray cascade processes of high energies, in this case an accurate investigation of backward scatterings is of decisive importance.

We note that the possibility of the way followed by us has already been

[11] L. JÁNOSSY: *Proc. Phys. Soc.*, A **63**, 241 (1950); L. JÁNOSSY and H. MESSEL: *Proc. Phys. Soc.*, A **63**, 1101 (1950).

pointed out by JÁNOSSY [12]; introducing certain generalizations [13] he even eliminated the difficulties referred to in the case of processes implying no change in the number of particles, and applies the results of this calculations for determining the energy distribution of particles slowing down in nuclear emulsions. The further generalization of the equation given by JÁNOSSY was dealt with by the autor [10] as well.

The other difference is due to the fact that, on the one hand, the cosmic ray cascade process is usually treated as one taking place in an infinite space, on the other, the processes in atomic reactors take place naturally in a medium of finite dimensions, and thus it should be given what happens to neutrons arriving at the boundary of the medium. Suppose that the reactor defines a convex region and the space surrounding this region is empty. The reflector and shield surrounding the active core of the reactor are of course considered part of the reactor.) If a neutron in such a system has arrived at the boundary, it will get out into the empty space from which it cannot return to the reactor any more. The boundary condition has to express this circumstance. In the course of the following it will be shown that the treatment applied here especially simplifies the formulation of the boundary condition.

Let us turn now to the determination of the stochastic state of the reactor. We choose as the origin of our co-ordinate system the geometrical centre of the region V_R which is filled with a multiplying and moderating medium. Denote by V_r the convex region around the end-point of the radius vector $\boldsymbol{r}$. Although no restrictions need be made on the dimensions of the region V_r, and this region may also include a part of the empty space surrounding the reactor, it is still advisable to consider the region V_r as part of the region V_R occupied by the reactor. The neutrons should be classified at a time t according to whether they are inside or outside the region V_r. (For sake of accuracy we note that the neutrons being at the time t just on the covering surface of the region V_r are also considered to be inside the region.) Denote the number of neutrons inside and outside the region V_r by m and m' respectively. The neutrons both inside and outside the region may be classified according to the following definitions [14]:

Type T_1 denotes neutrons of energy lower than E, with directions of velocity lying in the cone $\Omega_{\boldsymbol{\Omega}}$ about $\boldsymbol{\Omega}$.

Type T_2 denotes neutrons of energies lower than E, with directions of velocity lying outside the cone $\Omega_{\boldsymbol{\Omega}}$ about $\boldsymbol{\Omega}$.

[12] L. JÁNOSSY: *Žu. Éxper. Teor.* **26**, 518, (1954); *Acta Phys. Hung.*, **2**, 289 (1953).
[13] L. JÁNOSSY: *Žu. Éxper. Teor.*, **30**, 351 (1956).
[14] V. A. SEVAST'JANOV: *Usp. Mat. Nauk*, **6**, 47 (1951).

Type T_3 denotes neutrons of energies higher than E with directions of velocity lying in the cone Ω_{Ω} about $\boldsymbol{\Omega}$; and finally

Type T_4 denotes neutrons of energies higher than E with directions of velocity lying outside the cone Ω_{Ω} about $\boldsymbol{\Omega}$.

Suppose that inside the region V_r the number of neutrons of type T_1 is m_1, the number of neutrons of type T_2 is m_2, of T_3 is m_3, and of T_4 is m_4 respectively. Denote further by the non-negative integers m_1', m_2', m_3' and m_4' the number of neutrons of the respective types outside the volume V_r.

In the following we describe the state of the system at a time t by introducing two four-dimensional vectors with non-negative integer components, i.e. the vectors

$$\boldsymbol{m} = (m_1, m_2, m_3, m_4) \tag{3.1}$$

and

$$\boldsymbol{m}' = (m_1', m_2', m_3', m_4') \tag{3.2}$$

which indicate that at the moment t the states

$$\mathfrak{S}_{\boldsymbol{m}} = m_1 T_1 + m_2 T_2 + m_3 T_3 + m_4 T_4 \tag{3.3}$$

and

$$\mathfrak{S}_{\boldsymbol{m}'} = m_1' T_1 + m_2' T_2 + m_3' T_3 + m_4' T_4 \tag{3.4}$$

exist. The state $\mathfrak{S}_{\boldsymbol{m}}$ actually expresses that inside the volume V_r at the time t the number of neutrons of type T_1 is m_1, that of type T_2 is m_2, etc. The meaning of $\mathfrak{S}_{\boldsymbol{m}'}$ is that outside the volume V_r at the moment t the number of neutrons of type T_1 is m_1', that of type T_2 is m_2', etc.

Denote by $H(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}')$ the probability that at the moment t the neutron distribution defined by the vectors $\boldsymbol{m}$ and $\boldsymbol{m}'$ is fulfilled, provided that at a time $t = 0$ a neutron of energy E_0 and direction of velocity $\boldsymbol{\Omega}_0$ was present in the point defined by the radius vector $\boldsymbol{r}_0$. Our task consists really in the determination of the distribution function $H(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}')$. It is evident that, if we know this distribution function, we may obtain in principle exact answers to any question of neutronics. Unfortunately, however, the determination of the distribution function $H(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}')$ has proved impossible in the majority of cases. Nevertheless, it is not so much the distribution function itself, as especially its first and possibly second momentum, that are needed. The momenta, however, can be most symply derived from the generating function $G(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{q}, \boldsymbol{q}')$ of the distribution function $H(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}')$.

4. – In the following we discuss the determination of the distribution function $H(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}')$. The first collision of a neutron of energy E_0 (denoted δ-neutron in the following) which starts at the moment $t = 0$ from the end-point of the radius vector $\boldsymbol{r}_0$ in direction $\boldsymbol{\Omega}_0$, results in either absorption or scattering (elastic or inelastic) or fission. A further possibility is that the neutron does not suffer collision in the time interval $(0, t)$ at all. Taking into account these four possible and exclusive starting events of the process, we construct the distribution function $H(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}')$. Thus

$$H = H_0 + H_a + H_s + H_f \,, \tag{4.1}$$

where the distribution functions on the right hand side represent the following in succession: H_0 is the probability that the neutron of energy E_0 starting at the moment $t = 0$ from the end point of the vector $\boldsymbol{r}_0$ suffers no collision in the interval $(0, t)$, and at the moment t the state of the system is defined by the vectors $\boldsymbol{m}$ and $\boldsymbol{m}'$. H_a, H_s and H_f are in succession the probabilities that at the first collision in the interval $(0, t)$ the δ-neutron is captured, scattered, and causes fission respectively, and at the time t the state of the system is defined by the vectors $\boldsymbol{m}$ and $\boldsymbol{m}'$.

We calculate now the distribution functions on the right hand side of the expression (4.1). Let $\Delta(V_R, P_{\boldsymbol{\rho}_0}) = 1$, if the end-point $P_{\boldsymbol{\rho}_0}$ of the vector $\boldsymbol{\rho}_0$ is inside the region V_R, and zero in any other case. It is convenient to generalize the expressions (2.1), (2.2) and (2.3) as follows:

$$w_s^*(\boldsymbol{\rho}_0, E_0, \boldsymbol{\Omega}_0; E, \boldsymbol{\Omega}) = w_s(\boldsymbol{\rho}_0, E_0, \boldsymbol{\Omega}_0; E, \boldsymbol{\Omega})\Delta(R_{R'}, P_{\boldsymbol{\rho}_0}) \tag{4.2}$$

and

$$Q_i^*(\boldsymbol{\rho}_0, E_0) = Q_i(\boldsymbol{\rho}_0, E_0)\Delta(R_R, P_{\boldsymbol{\rho}_0}) \qquad (i = s, a, f, 0) \,, \tag{4.2'}$$

We define now the probability that the neutron of energy E_0 starting at the moment $t = 0$ from the end-point of the vector $\boldsymbol{r}_0$ in the direction $\boldsymbol{\Omega}_0$ suffers no collision in the interval $(0, t)$. Denote this probability by $F(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)$. Obviously we have

$$F(t + \Delta t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) = F(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)\{1 - v_0 Q_0^*(\boldsymbol{r}_0 + v_0 t\boldsymbol{\Omega}_0; E_0)\Delta t\} + 0(\Delta t)$$

from this we get

$$F(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) = \exp\left[-v_0 \int_0^t Q_0^*[\boldsymbol{\rho}_0(t'), E_0]\,\mathrm{d}t'\right] , \tag{4.3}$$

where

$$\boldsymbol{\rho}_0(t') = \boldsymbol{r}_0 + v_0 t'\boldsymbol{\Omega}_0 \,.$$

4·1. – Let us proceed now to determine the distribution function $H(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}')$. We introduce the following functions: Let $\Delta(V_r, P_{\boldsymbol{\rho}_0}) = 1$, if the end-point $P_{\boldsymbol{\rho}_0}$ of the vector $\boldsymbol{\rho}_0$ is inside the region V_r, and let is be equal zero in any other case. Further, let $\Delta(\Omega_{\boldsymbol{\Omega}}, \boldsymbol{\Omega}_0) = 1$, if the vector $\boldsymbol{\Omega}_0$ lies in the cone $\Omega_{\boldsymbol{\Omega}}$ about the vector $\boldsymbol{\Omega}$, and equal zero if the vector $\boldsymbol{\Omega}_0$ is outside this cone. Finally we put $\Delta(E, E_0) = 1$ for $E \geqslant E_0$ and $\Delta(E, E_0) = 0$ for $E < E_0$. Denote the quantities $\Delta^{-1}(V_r, P_{\boldsymbol{\rho}_0})$, $\Delta^{-1}(\Omega_{\boldsymbol{\Omega}}, \boldsymbol{\Omega}_0)$ and $\Delta^{-1}(E, E_0)$ as the inverse of the above quantities. Let us denote by γ_i the probability that a γ-neutron is of type T_i. It can be shown easily that

$$(4.5) \qquad \begin{cases} \gamma_1 = \Delta\,(\Omega_{\boldsymbol{\Omega}}, \boldsymbol{\Omega}_0)\,\Delta(E, E_0)\,, \quad \gamma_2 = \Delta^{-1}(\Omega_{\boldsymbol{\Omega}}, \boldsymbol{\Omega}_0)\,\Delta(E, E_0)\,, \\ \gamma_3 = \Delta(\Omega_{\boldsymbol{\Omega}}, \boldsymbol{\Omega}_0)\,\Delta^{-1}(E, E_0)\,, \quad \gamma_4 = \Delta^{-1}(\Omega_{\boldsymbol{\Omega}}, \boldsymbol{\Omega}_0)\,\Delta^{-1}(E, E_0)\,. \end{cases}$$

After introducing the generalized Kronecker symbols $\delta_{0\boldsymbol{m}} = \prod_{i=1}^{4} \delta_{0m_i}$ and $\delta_{0\boldsymbol{m}'} = \prod_{i=1}^{4} \delta_{0m_i'}$ we may easily construct the strongly singular distribution function H_0. We find that

$$(4.6) \qquad H_0(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}') =$$

$$= \delta_{0\boldsymbol{m}}\delta_{0\boldsymbol{m}'}\left\{\Delta\,(V_r, P_{\boldsymbol{\rho}_0(t)}) \sum_{i=1}^{4} \gamma_i \frac{\delta_{1m_i}}{\delta_{0m_i}} + \Delta^{-1}(V_r, P_{\boldsymbol{\rho}_0(t)}) \sum_{i=1}^{4} \gamma_i \frac{\delta_{1m_i'}}{\delta_{0m_i'}}\right\} F(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)\,,$$

where

$$\boldsymbol{\rho}_0(t) = \boldsymbol{r}_0 + v_0 t \boldsymbol{\Omega}_0\,.$$

4·2. – In the case when at the first collision the δ-neutron is captured, we can write the following expression:

$$(4.7) \qquad H_a(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}') = \delta_{0\boldsymbol{m}}\delta_{0\boldsymbol{m}'} v_0 \int_0^t Q_a^*[\boldsymbol{\rho}_0(t'), E_0] F(t', \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)\,\mathrm{d}t'\,.$$

4·3. – If the neutron is scattered at the first collision then, after the scattering, the neutron if it changed energy may initiate in the remaining period a similar process to that caused by the δ-neutron. Suppose that in the period $(0, t')$ no collision occurs, further in the time interval $(t', t' + \mathrm{d}t')$ a scattering takes place, as a result of which the energy of the δ-neutron is reduced from E_0 to a value inside the interval $(E', E' + \mathrm{d}E')$, and the initial direction $\boldsymbol{\Omega}_0$ of its velocity is changed into the direction inside the cone $\mathrm{d}\Omega'$ about $\boldsymbol{\Omega}'$. During the time $t - t'$ still available the scattered neutron initiates a process which at the moment t results in the neutron distribution characterized by the vectors $\boldsymbol{m}$ and $\boldsymbol{m}'$. Taking into account all possible cases of the moment of collision,

the changes in energy and in direction of velocity, we find

$$(4.8)\qquad H_s(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}') = v_0 \int_0^t \int_0^\infty \int F(t', \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)\, Q[\boldsymbol{\rho}_0(t'), E_0] \cdot$$

$$\cdot w_s[\boldsymbol{\rho}_0(t'), E_0, \boldsymbol{\Omega}_0; E', \boldsymbol{\Omega}']\, H[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}', E'; \boldsymbol{m}, \boldsymbol{m}']\, \mathrm{d}t'\, \mathrm{d}E'\, \mathrm{d}\Omega'.$$

Considering the thermal agitation of the atoms in the moderating medium, it is to be expected that, in particular for small values of the energy E_0, at the collision the energy of the δ-neutron does not decrease, but increases. The probability density function w_s^* in the expression (4.8) was accordingly chosen.

4·4. – Finally, in the case when the first collision leads to a fission, the further development of the process is determined by the forming of new « branches » initiated by prompt and delayed neutrons which were produced in the fission. Suppose that in the time interval $(0, t')$ no collision occurs, and in the subsequent time interval $(t', t'+\mathrm{d}t')$ a fission takes place resulting in k_0 prompt neutrons and $k_1, \ldots, k_l$ fission products. The prompt neutrons and the delayed neutrons emitted by the fission products initiate further processes and in the period $t-t'$ left they produce the neutron distribution defined by the vectors $\boldsymbol{m}$ and $\boldsymbol{m}'$. We put $K[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}_0, E_0, k_0; \boldsymbol{m}_0, \boldsymbol{m}_0']$ for the probability that in the period $t-t'$ the k_0 prompt neutrons, arising from the fission which occured at a place determined by the vector $\boldsymbol{\rho}_0(t')$, produce the neutron distribution defined by the vectors $\boldsymbol{m}_0$ and $\boldsymbol{m}_0'$, provided the energy and the direction of velocity of the neutron initiating the fission was respectively E_0 and $\boldsymbol{\Omega}_0$. Similarly we denote by $K[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}_0, E_0, k_i; \boldsymbol{m}_i, \boldsymbol{m}_i']$ the probability that in the period $t-t'$ the delayed neutrons emitted by the k_i fission products of type i which were produced in the fission taking place at the end point of the vector $\boldsymbol{\rho}_0(t')$, give rise to the neutron distribution defined by the vectors $\boldsymbol{m}_i$ and $\boldsymbol{m}_i'$, provided again that the energy and direction of velocity of the neutron initiating the fission was E_0 and $\boldsymbol{\Omega}_0$ respectively. With the help of the distribution function the distribution function H_f can be easily expressed, namely

$$(4.9)\qquad H_f[t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}') = v_0 \int_0^t F(t', \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) Q^*[\boldsymbol{\rho}_0(t'), E_0] \cdot$$

$$\cdot \Big\{ \sum_{k_0, \ldots, k_l} p(E_0, k_0, \ldots, k_l) \sum_{\boldsymbol{m}=\boldsymbol{m}_0+\ldots+\boldsymbol{m}_l} \sum_{\boldsymbol{m}_0'+\ldots+\boldsymbol{m}_l'=\boldsymbol{m}'} \prod_{i=0}^{l} K[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}_0, E_0, k_i; \boldsymbol{m}_i, \boldsymbol{m}_i'] \Big\}\, \mathrm{d}t'\,.$$

It can be easily shown that the distribution function $K[t-t', \boldsymbol{\rho}_0(t)', \boldsymbol{\Omega}_0, E_0, k_0; \boldsymbol{m}_0, \boldsymbol{m}_0']$ may be reduced to the distribution function $H(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}')$

describing the process initiated by the δ-neutron. Namely as each prompt neutron initiates a process independently of the others, we can write

$$(4.10)\qquad K[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}_0, E_0, k_0; \boldsymbol{m}_0, \boldsymbol{m}_0'] =$$

$$= \sum_{\boldsymbol{m}_0=\boldsymbol{m}_1+\ldots+\boldsymbol{m}_k} \sum_{\boldsymbol{m}_1'+\ldots+\boldsymbol{m}_{k_0}'=\boldsymbol{m}_0'} \prod_{j=1}^{k_0} \int_0^\infty\int w_f^0(E_0, \boldsymbol{\Omega}_0; E', \boldsymbol{\Omega}')\cdot$$

$$\cdot H[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}', E'; \boldsymbol{m}_j, \boldsymbol{m}_j'] \mathrm{d}E' \mathrm{d}\Omega' .$$

We can determine the distribution function $K[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}_0, E_0, k_i; \boldsymbol{m}_i, \boldsymbol{m}_i']$ $(i=1, 2, \ldots, l)$ as follows. In general not all of the k_i fission products suffer decay during the period $t-t'$. Suppose that during the time $t-t'$ a part k_i' of the fission products suffers decay, whereas $k_i'' = k_i - k_i'$ does not. At a small distance from the place of the fission and with certain delay as compared to the moment of fission the neutrons arising from the decay initiate a process similar to that initiated by the first δ-neutron injected into the system at the time $t=0$. We denote by $K_i[t-t', \boldsymbol{\rho}_0(t), \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}_j, \boldsymbol{m}_j']$ the probability that one of the k_i fission products, say the j-th, suffers decay in the time interval (t', t), and out of a point of the region around the end-point of the vector $\boldsymbol{\rho}_0(t)$, which can practically be regarded infinitely small, the neutron brought about by the decay initiates a process as a result of which a neutron distribution defined by the vectors $\boldsymbol{m}_j$ and $\boldsymbol{m}_j'$ appears during the time $t-t'$, assuming again that the initial neutron has the energy E_0 and direction of velocity $\boldsymbol{\Omega}_0$. With the help of the distribution function K_i we can write now

$$(4.11)\qquad K[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}_0, E_0, k_i; \boldsymbol{m}_i, \boldsymbol{m}_i'] = \sum_{k_i'=0}^{k_i} \binom{k_i}{k_i'} \exp\left[-(k_i-k_i')(t-t')\right]\cdot$$

$$\cdot \sum_{\boldsymbol{m}_i=\boldsymbol{m}_1+\ldots+\boldsymbol{m}_{k_i'}} \sum_{\boldsymbol{m}_1'+\ldots+\boldsymbol{m}_{k_i'}'=\boldsymbol{m}_i'} \prod_{i=1}^{k_i'} K_i[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}_j \boldsymbol{m}_j'] .$$

Since the fission products recede into a very small distance only from the source of the fission, it can be assumed that the delayed neutrons start the new cycle (of course, with a time lag) practically from the spot of the fission act. Thus

$$(4.12)\qquad K_i[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}_j, \boldsymbol{m}_j'] = \int_0^{t-t'} \int_0^\infty\int \lambda_i \exp\left[-t''\lambda_i\right] w_f^i(E_0, \boldsymbol{\Omega}_0; E', \boldsymbol{\Omega}')\cdot$$

$$\cdot H[t-t'-t'', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}', E'; \boldsymbol{m}_j, \boldsymbol{m}_j'] \mathrm{d}t'' \mathrm{d}E' \mathrm{d}\Omega' .$$

With the help of a somewhat more careful procedure the movements of the fission products may also be considered. This is, however, not specially needed.

In this way the distribution function $H(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}')$ has been determined which satisfies the non-linear integral equation (4.1). It seems to be practically not feasible to find the general solution for equation (4.1). However, as has been referred to, for practical purposes it is not so much the distribution function itself, as rather its momenta which are required. The following task should thus consist in determining the generating function of the distribution function.

5. – We choose two four-dimensional vectors, denoting the first by $\boldsymbol{q} = (q_1, q_2, q_3, q_4)$ and the second by $\boldsymbol{q}' = (q_1', q_2', q_3', q_4')$; we construct the scalar products

$$\left\{ \begin{array}{l} (\boldsymbol{m} \cdot \boldsymbol{q}) \;\; = m_1 q_1 + m_2 q_2 + m_3 q_3 + m_4 q_4 \,, \\ (\boldsymbol{m}' \cdot \boldsymbol{q}') = m_1' q_1' + m_2' q_2' + m_3' q_3' + m_4' q_4' \,. \end{array} \right. \tag{5.1}$$

The generating function of the distribution function is given by the infinite series

$$G(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{q}, \boldsymbol{q}') = \sum_{m, m'} \exp\left[(\boldsymbol{m} \cdot \boldsymbol{q}) + (\boldsymbol{m}' \cdot \boldsymbol{q}')\right] H(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{m}, \boldsymbol{m}'] \,. \tag{5.2}$$

First of all we have evidently

$$G(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{O}, \boldsymbol{O}) \equiv 1 \,. \tag{5.3}$$

We are interested in particular in one of the special cases of the generating function; this can be derived by putting $q_1 = q$, $q_i = 0$ $(i = 2, 3, 4)$ and $q_i' = 0$ $(i = 1, 2, 3, 4)$. Introducing the notation

$$G(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; q, 0) = g(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; q) \tag{5.4}$$

the first derivative with respect to q of the generating function $g(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; q)$ gives for $q = 0$ a special, but for our purpose important first momentum of the distribution function. The equation

$$\langle m_1 \rangle = \left(\frac{\partial g(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; q)}{\partial q} \right)_{q=1} = M_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; V_r, \boldsymbol{\Omega}_\Omega, E) \,, \tag{5.5}$$

gives obviously the average number of those neutrons which at the moment t are inside the region V_r around the end-point of the vector $\boldsymbol{r}$, and which have vectors of velocity inside the cone Ω_Ω around the direction $\boldsymbol{\Omega}$ with energies smaller than E, provided that at the time $t = 0$ one single neutron of energy

E_0 and direction of velocity $\boldsymbol{\Omega}_0$ was present at the end-point of the vector $\boldsymbol{r}_0$.

The average neutron density $n_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{r}, \boldsymbol{\Omega}, E)$ which from a practical point of view is of more importance, can be easily determined from the function $M_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; V_r, \Omega_{\boldsymbol{\Omega}}, E)$, as

$$(5.5') \qquad M_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; V_r, \boldsymbol{\Omega}_{\boldsymbol{\Omega}}^*, E) = \int_{V_r} \int_{\Omega_{\boldsymbol{\Omega}}} \int_0^E n_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{r}', \boldsymbol{\Omega}', E') \, \mathrm{d}V_r \, \mathrm{d}\boldsymbol{\Omega}' \, \mathrm{d}E' .$$

$n_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{r}, \boldsymbol{\Omega}, E) \, \mathrm{d}V_r \, \mathrm{d}\Omega \, \mathrm{d}E$ is evidently equal to the average number of neutrons being at the moment t in the volume element $\mathrm{d}V_r$ aroung the end-point of the vector $\boldsymbol{r}$, with vectors of velocity lying inside the cone $\mathrm{d}\Omega$ around the direction $\boldsymbol{\Omega}$, and with energies inside the interval $(E, E+\mathrm{d}E)$, provided all neutrons originate from that single neutron of energy E_0 and direction of velocity $\boldsymbol{\Omega}_0$ which at the time $t = 0$ from the point defined by the vector $\boldsymbol{r}_0$ initiated the whole process.

In the course of the following it will be shown that in the integro-differential equation for determining $M_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; V_r, \Omega_{\boldsymbol{\Omega}}, E)$ no operation refers to the quantities V_r, $\Omega_{\boldsymbol{\Omega}}$, E. Thus for the sake of convenience, in the following the notation of these quantities will be omitted, their role showing itself namely in the initial condition

$$(5.6) \qquad \lim_{t \to 0} M_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; V_r, \Omega_{\boldsymbol{\Omega}}, E) = \Delta(V_r, P_{r_0}) \Delta(\Omega_{\boldsymbol{\Omega}}, \boldsymbol{\Omega}_0) \Delta(E, E_0)$$

only.

So as to determine the fluctuation

$$(5.7) \qquad D^2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) = M_2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) - M_1^2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)$$

the second momentum which can be derived on the basis of the relation

$$(5.8) \qquad \langle m_1^2 \rangle = \left(\frac{\partial^2 g(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; q)}{\partial q^2} \right)_{q=0} = M_2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; V_r, \boldsymbol{\Omega}_{\boldsymbol{\Omega}}, E) =$$

$$= \int_{V_r} \int_{\Omega_{\boldsymbol{\Omega}}} \int_0^E n_2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{r}', \boldsymbol{\Omega}', E') \, \mathrm{d}V\boldsymbol{r}_r \, \mathrm{d}\Omega' \, \mathrm{d}E' ,$$

is required. After constructing the generating function we write the equations for the functions M_1 and M_2, by which both the average neutron density n_1, and the mean square of the neutron density n_2 may be readily determined with the help of (5.5) and (5.8). The fluctuation of the neutron density is

given as in (5.7) by the expression

$$d^2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) = n_2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) - n_1^2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)\,. \tag{5.7'}$$

Before determining the momenta M_1 and M_2 we have to make following two remarks.

Denote by $Z(\boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)\,\mathrm{d}V_{r_0}\,\mathrm{d}\Omega_0\,\mathrm{d}E_0$ the probability that in the volume $\mathrm{d}V_{r_0}$ around the end-point of the vector $\boldsymbol{r}_0$ there is a neutron with energy lying inside the interval $(E_0, E_0+\mathrm{d}E_0)$ and direction of velocity in the cone $\mathrm{d}\Omega_0$ about $\boldsymbol{\Omega}_0$. If $Z(\boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)$ is known, the neutron density

$$n_1(t, \boldsymbol{r}, \boldsymbol{\Omega}, E) = \int\limits_{V_R}\int\limits_0^\infty\int Z(\boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)\, n_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0;\, \boldsymbol{r}, \boldsymbol{\Omega}, E)\,\mathrm{d}V_{r_0}\,\mathrm{d}\Omega_0\,\mathrm{d}E_0\,, \tag{5.9}$$

which is valid for arbitrary initial distribution, can be easily calculated.

The neutron importance, on the other hand, introduced by USSACHOFF [6] is given by

$$J(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) = \int\limits_{V_R}\int\limits_0^\infty\int n_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0;\, \boldsymbol{r}, \boldsymbol{\Omega}, E)\,\mathrm{d}V_r\,\mathrm{d}E\,\mathrm{d}\Omega\,. \tag{5.10}$$

According to the instruction under (5.2) the equation satisfied by the generating function $G(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{q}, \boldsymbol{q}')$ can easily be obtained. After simple calculation we find

$$G = G_0 + G_a + G_s + G_f\,, \tag{5.11}$$

where

$$G_0(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{q}, \boldsymbol{q}') = \tag{5.12}$$
$$= F(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)\Big\{\Delta(V_r, P_{\boldsymbol{r}_0(t)}) \sum_{i=1}^{4} \gamma_i e^{q_i} + \Delta^{-1}(V_r, P_{\boldsymbol{r}_0(t)}) \sum_{i=1}^{4} \gamma_i e^{q_i'}\Big\}$$

and

$$G_a(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{q}, \boldsymbol{q}') = v_0 \int\limits_0^t Q_a^*[\boldsymbol{\rho}_0(t'), E_0] F(t', \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)\,\mathrm{d}t'\,; \tag{5.13}$$

further

$$G_s(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; \boldsymbol{q}, \boldsymbol{q}') = v_0 \int\limits_0^t\int\limits_0^\infty\int F(t', \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) Q_s^*[\boldsymbol{\rho}_0(t'), E_0] \cdot \tag{5.14}$$
$$\cdot w_s^*[\boldsymbol{\rho}_0(t'), E_0, \boldsymbol{\Omega}_0;\, E', \boldsymbol{\Omega}']\, G[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}', E';\, \boldsymbol{q}, \boldsymbol{q}']\,\mathrm{d}t'\,\mathrm{d}E'\,\mathrm{d}\Omega'$$

and finally

$$(5.15)\quad G_f(t,\boldsymbol{r}_0,\boldsymbol{\Omega}_0,E_0;\boldsymbol{q},\boldsymbol{q}')=v_0\int_0^t F(t',\boldsymbol{r}_0,\boldsymbol{\Omega}_0,E_0)Q_f^*[\boldsymbol{\rho}_0(t'),E_0]\Big\{\sum_{k_0,\ldots,k_l}p(E_0,k_0,\ldots,k_l)\cdot$$

$$\cdot G_f^p[t-t',\boldsymbol{\rho}_0(t'),\boldsymbol{\Omega}_0,E_0,k_0;\boldsymbol{q},\boldsymbol{q}']\prod_{i=1}^{l}G_f^d[t-t',\boldsymbol{\rho}_0(t'),\boldsymbol{\Omega}_0,E_0,k_i;\boldsymbol{q},\boldsymbol{q}']\Big\}\,\mathrm{d}t'\,.$$

The functions G_f^p and G_f^d can be reduced in one or two steps to the function G, namely,

$$(5.16)\quad G_f^p=\Big\{\int_0^\infty\!\!\int w_f^0(E_0,\boldsymbol{\Omega}_0;E',\boldsymbol{\Omega}')\,G[t-t',\boldsymbol{\rho}_0(t'),\boldsymbol{\Omega}',E';\boldsymbol{q},\boldsymbol{q}']\,\mathrm{d}E'\,\mathrm{d}\Omega'\Big\}^{k_0}$$

and

$$(5.17)\quad G_f^d=\big\{G_f^i(t-t',\boldsymbol{\rho}_0(t'),\boldsymbol{\Omega}_0,E_0;\boldsymbol{q},\boldsymbol{q}']+\exp\left[-(t-t')\lambda_i\right]\big\}^{k_l},$$

where

$$(5.18)\quad G_f^i[t-t',\boldsymbol{\rho}_0(t'),\boldsymbol{\Omega}_0,E_0;\boldsymbol{q},\boldsymbol{q}']=\lambda_i\int_0^{t-t'}\!\!\int_0^\infty\!\!\int\exp\left[-t''\lambda_i\right]w_f^i(E_0,\boldsymbol{\Omega}_0;E',\boldsymbol{\Omega}')\cdot$$

$$\cdot G[t-t'-t'',\boldsymbol{\rho}_0(t'),\boldsymbol{\Omega}',E';\boldsymbol{q},\boldsymbol{q}']\,\mathrm{d}t''\,\mathrm{d}E'\,\mathrm{d}\Omega'\,.$$

According to the relations (5.4), (5.5) and (5.11) the equation for the first momentum of the distribution function can be written as follows:

$$(5.19)\quad M_1(t,\boldsymbol{r}_0,\boldsymbol{\Omega}_0,E_0)=M_c(t,\boldsymbol{r}_0,\boldsymbol{\Omega}_0,E_0)+v_0\int_0^t\!\!\int_0^\infty\!\!\int Q_s^*[\boldsymbol{\rho}_0(t'),E_0]F(t',\boldsymbol{r}_0,\boldsymbol{\Omega}_0,E_0)\cdot$$

$$\cdot w_s^*[\boldsymbol{\rho}_0(t'),E_0,\boldsymbol{\Omega}_0;E',\boldsymbol{\Omega}']M_1[t-t',\boldsymbol{\rho}_0(t'),\boldsymbol{\Omega}',E']\,\mathrm{d}t'\,\mathrm{d}E'\,\mathrm{d}\Omega'+$$

$$+v_0\int_0^t Q_f^*[\boldsymbol{\rho}_0(t'),E_0]F(t',\boldsymbol{r}_0,\boldsymbol{\Omega}_0,E_0)\sum_{i=0}^{l}r_iA_i[t-t',\boldsymbol{\rho}_0(t'),\boldsymbol{\Omega}_0,E_0]\mathrm{d}t'\,,$$

where

$$(5.19')\quad M_0(t,\boldsymbol{r}_0,\boldsymbol{\Omega}_0,E_0)=\Delta(V_r,P_{\boldsymbol{\rho}_0(t)})\Delta(\Omega_{\boldsymbol{\Omega}},\boldsymbol{\Omega}_0)\Delta(E,E_0)F(t,\boldsymbol{r}_0,\boldsymbol{\Omega}_0,E_0)$$

and

$$(5.20)\quad A_c[t-t',\boldsymbol{\rho}_0(t'),\boldsymbol{\Omega}_0,E_0]=\int_0^\infty\!\!\int w_f^0(E_0,\boldsymbol{\Omega}_0;E',\boldsymbol{\Omega}')M_1[t-t',\boldsymbol{\rho}_0(t'),\boldsymbol{\Omega}',E']\,\mathrm{d}E'\,\mathrm{d}\Omega'\,,$$

whereas for $i=1,2,\ldots,l$

$$(5.20')\quad A_i[t-t',\boldsymbol{\rho}_0(t'),\boldsymbol{\Omega}_0,E_0]=\int_0^{t-t'}\!\!\int_0^\infty\!\!\int\lambda_i\exp\left[-t''\lambda_i\right)w_f^i(E_0,\boldsymbol{\Omega}_0;E',\boldsymbol{\Omega}')\cdot$$

$$\cdot M_1[t-t'-t'',\boldsymbol{\rho}_0(t'),\boldsymbol{\Omega}',E']\,\mathrm{d}t''\,\mathrm{d}E'\,\mathrm{d}\Omega'\,.$$

For the second momentum of the distribution function the equation

$$
\begin{aligned}
(5.21)\qquad M_2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) &= M_0(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) + v_0 \int_0^t \iint_0^\infty Q_s^*[\boldsymbol{\rho}_0(t'), E_0] F(t', \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) \cdot \\
&\cdot w_s^*[\boldsymbol{\rho}_0(t'), E_0, \boldsymbol{\Omega}_0; E', \boldsymbol{\Omega}'] M_2[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}_0, E'] \, \mathrm{d}t' \, \mathrm{d}E' \, \mathrm{d}\Omega' + \\
&+ v_0 \int_0^t Q_f^*[\boldsymbol{\rho}_0(t'), E_0] F(t', \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) \sum_{i=0}^{l} \nu_i B_i[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}_0, E_0] \, \mathrm{d}t' + \\
&+ v_0 \int_0^t Q_f^*[\boldsymbol{\rho}_0(t'), E_0] F(t', \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) C[t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}_0, E_0] \, \mathrm{d}t' \, ,
\end{aligned}
$$

can be derived, where B_i $(i = 0, 1, ..., l)$ is identical to the expression A_i $(i = 0, 1, ..., l)$ of Eq. (5.20) only M_1 has to be replaced by M_2, while

$$
(5.22)\qquad C = \sum_{i=0}^{l} (\mu_i - \nu_i) A_i^2 + 2 \sum_{i \neq i'} \nu_i \nu_{i'} A_i A_{i'} \, .
$$

Eqs. (5.21) and (5.19) are nearly of the same construction. The only difference is due to the last term representing inhomogeneity. Eqs. (5.19) and (5.21) are often favourable in determining M_1 and M_2, being suitable for iteration.

Finally, let us derive the transport equations from Eqs. (5.19) and (5.21). Making use of the fact that $F(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) = F(t', \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) F(t-t', \boldsymbol{\rho}_0(t'), \boldsymbol{\Omega}_0, E_0)$ and carrying out some simple transformations, we get from (5.19)

$$
\begin{aligned}
(5.23)\qquad &\frac{\partial M_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)}{\partial t} + Q_0^*(\boldsymbol{r}_0, E_0) v_0 M_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) = \\
&= v_0 \boldsymbol{\Omega}_0 \cdot \operatorname{grad}_{\boldsymbol{r}_0} M_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) + v_0 Q_s^*(\boldsymbol{r}_0, E_0) \int_0^t \iint w_s^*(\boldsymbol{r}_0, E_0, \boldsymbol{\Omega}_0; E', \boldsymbol{\Omega}') \cdot \\
&\cdot M_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}', E') \, \mathrm{d}E' \, \mathrm{d}\Omega' + v_0 Q_f^*(\boldsymbol{r}_0, E_0) \sum_{i=0}^{l} \nu_i A_i(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) \, .
\end{aligned}
$$

With a similar procedure we obtain from (5.21) the integro-differential equation

$$
\begin{aligned}
(5.24)\qquad &\frac{\partial M_2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)}{\partial t} + Q_0^*(\boldsymbol{r}_0, E_0) v_0 M_2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) = \\
&= v_0 \boldsymbol{\Omega}_0 \cdot \operatorname{grad}_{\boldsymbol{r}_0} M_2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) + v_0 Q_s^*(\boldsymbol{r}_0, E_0) \int_0^\infty \iint w_s^*(\boldsymbol{r}_0, E_0, \boldsymbol{\Omega}_0; E', \boldsymbol{\Omega}') \cdot \\
&\cdot M_2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}', E') \, \mathrm{d}E' \, \mathrm{d}\Omega' + v_0 Q_f^*(\boldsymbol{r}_0, E_0) \sum_{i=0}^{l} \nu_i B_i(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) + \\
&+ v_0 Q_f^*(\boldsymbol{r}_0, E_0) C(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) \, .
\end{aligned}
$$

With the help of Eqs. (5.19) and (5.21) the initial condition of (5.23) and (5.24) can be easily written:

$$\lim_{t\to 0} M_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) = \lim_{t\to 0} M_2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) = \lim_{t\to 0} M_0(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0) = \tag{5.25}$$
$$= \Delta(V_r, P_{r_0})\Delta(\Omega_{\Omega}, \boldsymbol{\Omega}_0)\Delta(E, E_0) .$$

So as to formulate the boundary conditions we may proceed as follows. If P_{r_0} is inside the region V_R the Eqs. (5.23) and (5.24) are valid, only Q_i^* $(i = a, s, f, 0)$ has to be replaced by Q_i $(i = a, s, f, 0)$. If P_{r_0} is outside the region V_R, on the other hand, according to the definition of the cross-section Q_i^* $(i = a, s, f, 0)$ the equation

$$\frac{\partial\varphi(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0)}{\partial t} = v_0\boldsymbol{\Omega}_0 \cdot \operatorname{grad}_{\boldsymbol{r}_0} \varphi(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0), \tag{5.26}$$

holds both for M_1 and M_2, as long as the neutron does not reach the boundary surface of the region V_R. The initial condition for Eq. (5.26) is equivalent to that for Eq. (5.25). Concerning the solution of Eq. (5.26) the following should be noted. Suppose that from the end-point P_{r_0} of the vector $\boldsymbol{r}_0$ the region V_R is viewed, inside the cone Ω_R (Fig. 1). If $\boldsymbol{\Omega}_0$ is outside this cone, the solution of (5.26) is given by

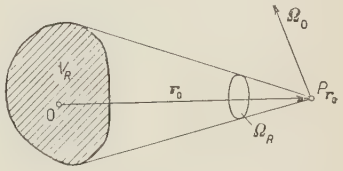

Fig. 1.

$$\varphi(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; V_r, \Omega_{\Omega}, E) = M_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; V_r, \Omega_{\Omega}, E) = \tag{5.27}$$
$$= M_2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; V_r, \Omega_{\Omega}, E) = \Delta(V_r, P_{r_0(t)})\Delta(\Omega_{\Omega}, \boldsymbol{\Omega}_0)\Delta(E, E_0) .$$

If $\boldsymbol{\Omega}_0$ is inside the cone Ω_R defined above, the following may be said. Let t_0 be the period during which the neutron of energy E_0 starting from the point P_{r_0} in the direction $\boldsymbol{\Omega}_0$ arrives at a point defined by the vector $\boldsymbol{r}_{s_0}$ on the boundary of the region V_R (Fig. 2). It is evident that in the time interval $0 \leqslant t \leqslant t_0$ the solution of Eq. (5.26) is identical with the expression (5.27), but for a $t > t_0$ already (5.23) and (5.24) hold. Denote by $\boldsymbol{\Omega}_s$ the normal belonging to the end-point of the vector $\boldsymbol{r}_s$ oriented outwards. Finally the boundary

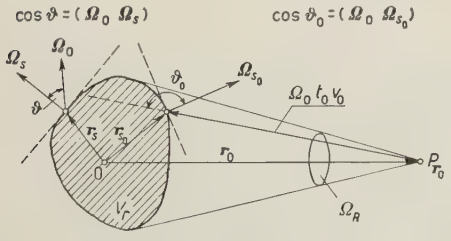

Fig. 2.

conditions may be written as follows

$$
(5.28)\quad
\begin{cases}
\lim\limits_{r_0 \to r_s} M_1(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; V_r, \Omega_{\boldsymbol{\Omega}}, E) = \begin{cases} \Delta(V_r, P_{\boldsymbol{\varrho}_0(t)})\,\Delta(\Omega_{\boldsymbol{\Omega}}, \boldsymbol{\Omega}_0)\,\Delta(I, \boldsymbol{\Omega}_0) \\ \qquad\qquad \text{for} \quad 0 \leqslant (\boldsymbol{\Omega}_0 \cdot \boldsymbol{\Omega}_s) \leqslant 1 \\ M_1(t, \boldsymbol{r}_s, \boldsymbol{\Omega}_0, E_0; V_r, \Omega_{\boldsymbol{\Omega}}, E) \\ \qquad\qquad \text{for} \quad -1 \leqslant (\boldsymbol{\Omega}_0 \cdot \boldsymbol{\Omega}_s) < 0 \end{cases} \\
\text{and} \\
\lim\limits_{r_0 \to r_s} M_2(t, \boldsymbol{r}_0, \boldsymbol{\Omega}_0, E_0; V_r, \Omega_{\boldsymbol{\Omega}}, E) = \begin{cases} \Delta(V_r, P_{\boldsymbol{\varrho}_0(t)})\,\Delta(\Omega_{\boldsymbol{\Omega}}, \boldsymbol{\Omega}_0)\,\Delta(E, E_0) \\ \qquad\qquad \text{for} \quad 0 \leqslant (\boldsymbol{\Omega}_0 \cdot \boldsymbol{\Omega}_s) \leqslant 1 \\ M_2(t, \boldsymbol{r}_s, \boldsymbol{\Omega}_0, E_0; V_r, \Omega_{\boldsymbol{\Omega}}, E) \\ \qquad\qquad \text{for} \quad -1 \leqslant (\boldsymbol{\Omega}_0 \cdot \boldsymbol{\Omega}_1) < 0\,. \end{cases}
\end{cases}
$$

This general theoretical treatment of stochastic processes taking place in reactors will be followed in a subsequent paper by some examples showing the application of the theory.

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The Theory of Electron Motion in Cyclic Accelerators in Presence of Radiation.

A. A. Kolomenskij and A. N. Lebedev

P. N. Lebedev Institute of Physics - Moscow, USSR

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1. – Introduction.

In recent years the problem of construction of powerful electronic synchrotrons with energies of the order 1 GeV and more has attracted great attention.

Machines of this type are destined for the production of and experiments with hard Bremsstrahlung γ-rays and for experimental work with high energy electron beams. The use of γ-rays, an electromagnetic radiation which has been extensively studied, greatly facilitates the performance and interpretation of experiments on elementary particles.

Great electronic synchrotrons are essentially characterized by the presence of intensive relativistic electromagnetic radiation of electrons in the magnetic field of the accelerator. The power of this radiation rapidly increases with the

increase of the electron energy E

$$W = \frac{2}{3}\frac{e^2 c}{r_c^2}\gamma^4 , \tag{1.1}$$

where r_c is radius of curvature of the orbit, and

$$\gamma = \frac{E}{mc^2} \gg 1 . \tag{1.1a}$$

Consequently, already at not very high electron energies the major part of the power of the accelerating system is expended on compensating for radiation losses. Thus at 10^9 eV the losses amount to $\simeq 20$ keV/r. An increase of energy merely by a factor of 10 (to $\simeq 10$ GeV) results in losses up to $20 \div 30$ MeV/r.

Another essential feature inherent to the radiation of relativistic particles in general is its sharp directivity: practically all the intensity is concentrated within a narrow cone of angle γ^{-1} about the direction of the instantaneous velocity of the electron. Calculations showed that within γ^{-2} all radiation may be assumed to be tangential to the trajectories of the electrons.

The radiation spectrum is very rich in higher harmonics of the fundamental frequency c/R. In contrast to the non-relativistic case the maximum of the radiation spectrum has the frequency

$$\omega_m = \frac{3}{2}\frac{c}{R}\gamma^3 , \tag{1.2}$$

which at high energies ($\gtrsim 1$ GeV) corresponds to the region of soft X-rays.

The foregoing considerations as well as formulas (1.1) and (1.2) are derived from the classical theory of radiation. To determine their limits of applicability the following simple criterion may be used [1]: the quantum corrections for the fundamental characteristics of the radiation are small if its wave-length considerably exceeds the de Broglie wave of the electron, or, which is the same, if the energy of the electron is much greater than the energy quantum $E \gg \hbar\omega_m$. Formally these conditions may be resumed as follows:

$$E \ll E_{\frac{1}{2}} = mc^2 \left[\frac{Rmc}{\hbar}\right]^{\frac{1}{2}} . \tag{1.3}$$

As seen from (1.3) the movement of electrons up to the highest electron energies possible in a synchrotron may be regarded as classical, while in ge-

[1] V. VLADIMIRSKIJ: *Žu. Éksper. Teor. Fiz.*, **18**, 392 (1948).

neral the effect of quantum (statistical) fluctuation of the radiation itself cannot be neglected ([2,3]). The fundamental character of this effect was first pointed out by SOKOLOV and TERNOV ([4]).

2. – Motion of electrons in an axially-symmetrical accelerator in presence of radiation.

Let us first consider the movement of a relativistic electron in an axially-symmetrical magnetic field under action of an external accelerating field taking into account the radiation.

In the present paragraph we shall discuss the results obtained within the frame-work of the classical theory, basing on the relativistic equation for the movement of a point electron ([5]).

$$mc\frac{\mathrm{d}U_i}{\mathrm{d}s} = \frac{e}{c}F_{ik}U_k + \frac{2e^2}{3c}\left[\frac{\mathrm{d}^2U_i}{\mathrm{d}s^2} + U_iU_k\frac{\mathrm{d}^2U_k}{\mathrm{d}s^2}\right], \tag{2.1}$$

where U_i are the components of the 4-dimensional velocity and F_{ik} the components of the field tensor. The quantity in brackets is the « self-acting » force, due to radiation. The Z-axis is directed along the magnetic field. Since we consider the movement about the equilibrium orbit with a constant radius R, we must introduce the electric field $\boldsymbol{\mathcal{E}}$, which compensates for energy losses. (We shall consider the familiar case of $\boldsymbol{\mathcal{E}}$ having only the azimuthal component $\mathcal{E}$). The field components have the following form:

$$\begin{cases} \mathcal{E}_x = -\dfrac{y}{r}\mathcal{E}; & H_z(r) = H_z(R)\left[1 - n\dfrac{r-R}{R}\right]; \\ \mathcal{E}_y = \dfrac{x}{r}\mathcal{E}; & H_r = n\dfrac{H_z(R)}{R}z; \qquad r^2 = x^2 + y^2\,. \end{cases} \tag{2.2}$$

In the case of a synchrotron $\mathcal{E}$ may be represented as a travelling wave (see for instance ref ([6])):

$$\mathcal{E} = \frac{V_0\cos\varphi}{2\pi r}, \tag{2.3}$$

([2]) A. KOLOMENSKIJ and A. LEBEDEV: *Žu. Éksper. Teor. Fiz.*, **30**, 205, 1161 (1956); *Symposium CERN*, Geneva (1956), p. 477.

([3]) M. SANDS: *Phys. Rev.*, **97**, 470 (1955).

([4]) A. SOKOLOV and I. TERNOV: *Žu. Éksper. Teor. Fiz.*, **24**, 249 (1953); **28**, 431 (1955); A. SOKOLOV: *Suppl. Nuovo Cimento*, **3**, 743 (1956).

([5]) L. LANDAU and E. LIFŠIC: *Theory of Field* (Moscow, 1948).

([6]) D. BOHM and L. FOLDY: *Phys. Rev.*, **70**, 249 (1946).

where V_0 is the total amplitude of the voltage at the accelerating gaps and φ the phase of the particle relative to voltage.

From (2.1)–(2.3) we obtain the linearized equations of motion in the following form:

$$\frac{1}{E_s}\frac{\mathrm{d}}{\mathrm{d}t}[E_s\dot{\varrho}] + \frac{W_s}{E_s}\dot{\varrho} + \frac{c^2}{R^2}(1-n)\varrho = \frac{c^2}{R}\frac{E-E_s}{E_s}, \tag{2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(E-E_s) = -\frac{ceV_0\sin\varphi_s}{2\pi R}\eta - W_s\left[(1-2n)\frac{\varrho}{R} + 2\frac{E-E_s}{E_s}\right], \tag{2.5}$$

$$\frac{1}{E_s}\frac{\mathrm{d}}{\mathrm{d}t}[E_s\dot{z}] + \frac{W_s}{E_s}\dot{z} + \frac{c^2}{R^2}nz = 0, \tag{2.6}$$

(where the index s denotes the equilibrium values). In addition, in the right-hand part of (2.4)–(2.6) appear quantities of the type $(W_s/E_s)(R^2/c^2\gamma^2)\ddot{\varrho}$ etc., describing the low « proper » radiation damping of the betatron oscillations. Since these terms are of the order γ^{-2} relative to the terms describing the fundamental radiation « friction », they can be ignored. To equation (2.4)–(2.6) must be added the linear relation which bounds the values $\eta = \varphi - \varphi_s$ and $\varrho = r - R$

$$\dot{\eta} = q\frac{c}{R^2}\varrho, \tag{2.7}$$

where q is the number of r.f. harmonics. From the system (2.4)–(2.6) we derive the equation of radial motion

$$\ddddot{\varrho} + \dddot{\varrho}\left[2\frac{\dot{E}_s}{E_s} + 3\frac{W_s}{E_s} - 2\frac{\dot{\Omega}}{\Omega}\right] + \frac{c^2}{R^2}(1-n)\dot{\varrho} + \\ + \dot{\varrho}\left[(1-n)\left(\frac{\dot{E}_s}{E_s} - 2\frac{\dot{\Omega}}{\Omega}\right) + (3-4n)\frac{W_s}{E_s}\right]\frac{c^2}{R^2} + \frac{c^2}{R^2}(1-n)\Omega^2\varrho = 0, \tag{2.8}$$

where Ω (within small corrections) is the frequency of the synchrotron oscillations

$$\Omega^2 = \frac{qc^2eV}{2\pi R^2(1-n)E_s}; \qquad V = V_0\sin\varphi_s. \tag{2.9}$$

2·1. *Vertical oscillations.* – Consider first the more simple equation of z-oscillations (2.6). It differs from the familiar equation of betatron oscillations (valid in absence of radiation) by the presence of the term $(W_s/E_s)\dot{z}$ which corresponds to the vertical component of the radiation reaction force.

In absence of compensation for radiation losses $\dot{E}_s = -W_s$ and the oscil-

lations amplitude a_z remains constant. But if, as is always the case in practice, the accelerating system not only compensates for radiation losses, but imparts to the electron the energy $\dot{E}_s = e\dot{H}(R)R$, the amplitude a_z according to (2.6) varies as

$$a_z \sim E_s^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\int_0^t \frac{W_s}{E_s}\,\mathrm{d}x\right]. \tag{2.10}$$

For the evaluation of the corresponding relaxation time we obtain the expression

$$T_{\mathrm{rel}} \simeq \frac{2\cdot 10^{-2}}{E_{\max}\,(\mathrm{GeV})}\ \mathrm{s}\,; \qquad a \sim \exp\left[-\frac{T_{ac}}{T_{\mathrm{rel}}}\right], \tag{2.11}$$

where T_{ac} is the time of acceleration of the particles up to the highest maximum energies.

It must be noted that even if the energy is imparted to the particle not continuously, but by discrete portions, in the accelerating gaps, radiation damping would practically be still described by the same formula (2.10).

2·2. *Radial oscillations.* – Now consider the radial oscillations ϱ, described by the equation (2.8). It is easily seen that in the zero approximation, in which terms of the order W_s/E_s $\dot{E}_s/E_s$, etc., are neglected, the solution of this expression is a superposition of rapid ϱ_b and slow ϱ_c (betatron and synchrotron) oscillations.

The solution of (2.8) which is a first approximation equation, will be sought in the form:

$$\varrho = a_{\varrho b}(t)\exp\left[i\,\frac{c}{R}\sqrt{1-n}\,t\right] + a_{\varrho c}(t)\exp\left[i\int_0^t \Omega\,\mathrm{d}\tau\right] + \mathrm{c.c.}\,, \tag{2.12}$$

where $a_{\varrho b}$, $a_{\varrho c}$ are slow functions of time, the first deviations of which are to be considered as being of the same order of smallness as W_s/E_s, $\dot{E}_s/E_s$, hereby $W_s/E_s \ll \Omega\cdot(c/R)\sqrt{1-n}$. This last assumption means that the radiation damping during a period of betatron and synchrotron oscillation is small, which is true up to $E \simeq 10^{11}$ eV. For the law of variation of the betatron oscillations amplitude the following results are obtained

$$a_{\varrho b} \sim E_s^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\,\frac{n}{1-n}\int_0^t \frac{W_s}{E_s}\,\mathrm{d}x\right], \tag{2.13}$$

while for synchrotron oscillations the expression obtained

$$a_{\varrho c} \sim V^{+\frac{1}{4}} E_s^{-\frac{3}{4}} \exp\left[-\frac{1}{2}\frac{3-4n}{1-n}\int_0^t \frac{W_s}{E_s}\,\mathrm{d}x\right], \tag{2.14}$$

coincides with the result of ref [3,6].

As seen from (2.13) and (2.14) the oscillations ϱ_b are subject to a powerful radiational friction. The essential difference between them and the above mentioned case of z-oscillations (see (2.10)) consists in the dependence of the sign and magnitude of friction on the value of the field index n.

If the condition

$$0 < n < 1 \tag{2.15}$$

characteristic for weak focusing is maintained, the damping of ϱ_b occurs. If this condition does not hold the antidamping occurs (*).

The damping of ϱ_c takes place in the intervals

$$a) \quad n < \tfrac{3}{4}\,, \qquad\qquad b) \quad n > 1\,. \tag{2.16}$$

The corresponding relaxation time in this cases has the same order of magnitude as for z-oscillations.

The radiation friction for radial betatron oscillation (see (2.13)) is a result of the superposition of several effects differing in character: *a*) the effect of the radial component of the radiation reaction force in presence of compensation of radiation losses leads to damping similar to the damping of vertical oscillations; *b*) the dependence of the power of radiation losses W (see (1.1)) on the value of H in a given point, $W \sim H^2(r)$, leads to a perturbation of $\dot{E}$ equal to $\Delta\dot{E} = 2nW_s(\varrho R/)$(see (2.5)); *c*) the effect linked with the fact that for a particle deviating by the amount ϱ the length of the orbit is increased as compared with the equilibrium orbit, while the power of compensation of the losses is not directly dependent on ϱ, leads to $\Delta\dot{E} = -W_s(\varrho/R)$.

Owing to effects *b*) and *c*) the radiation losses and their compensation depend on the phase of betatron oscillations and are subject to rapid variation occurring at the frequency of this oscillations.

The simultaneous effect of *b*) and *c*) may be qualitatively explained if ϱ_b is considered as the oscillation of a damped pendulum under action of a quasi-elastic force $f_\varrho \sim -(1-n)(\varrho/R)$. According to the above, the equilibrium point of the pendulum, the position of which is characterized by $\Delta E = E - E_s$, itself oscillates and is shifted with a velocity $\Delta\dot{E} \sim (2n-1)W_s(\varrho/R)$. This

(*) Antidamping of this type and methods of suppression are discussed in Sect. **4**.

may lead either to damping if $\Delta\dot{E}$ and j_ϱ have opposite signs, or to antidamping if this condition does not hold.

In conjunction with the effect *a*), it leads to damping conditions (2.15).

3. - Consideration of radiation fluctuations during the motion of electrons in an axially-symmetric accelerator.

As noted in Sect. **1**, to take into account quantum effects in the present case means that on the basis of the classical theory we must additionally take into account the corrections due to statistical fluctuation of radiation. We make now the following assumptions, valid for $E \ll E_{\frac{1}{2}}$.

a) The emission of individual photons on the equilibrium orbit is statistically independent, i.e. obeys the Poisson law. The correlation correction is not essential as it can be easily seen from (1.1), (1.2) according to which $\hbar\omega_m/E \ll 1$.

b) The emission of photons occurs instantaneously. The uncertainty of the moment of emission is of the order of $\delta t \simeq 1/\omega_m \approx R/c\gamma^3$, i.e. is negligibly small as compared with any of the characteristic time scales of our problem.

c) Spectrum and intensity are derived on the basis of the classical theory and are valid up to terms of the order $[E/E_{\frac{1}{2}}]^2$.

The validity of this treatment in the region $E \ll E_{\frac{1}{2}}$, i.e. at $\hbar\omega_m \ll E$ follows from the general assumptions of the radiation theory of quantum electrodynamics (see for instance ref. [7]).

As an illustration it can be noted that in a 5 GeV accelerator the electron moving at the highest energy emits $\simeq 100$ photons/r with a wave-length $\lambda\!\!\!^{-}_m \simeq 1$ Å (radius of the orbit $\simeq 15$ m), the ratio $E/E_{\frac{1}{2}}$ being $\simeq 10^{-3}$.

It follows from the foregoing that in order to account for the statistical character of radiation losses, the equation (2.5) for $\dot{E}$ must be completed by introducing

$$w(t) = \sum_i \varepsilon_i \delta(t - t_i)\,, \tag{3.1}$$

where $\varepsilon_i = \hbar\omega_i$ is the energy of the i-th photon, and δ the delta-function. By taking into account $w(t)$ the system of equations (2.4), (2.6) is turned into an inhomogeneous system and accordingly (2.8) becomes an inhomogeneous equation with the rigth-hand part $(c^2/R)(w/E_s)$. Further changes in (2.8) connected with the consideration of $w(t)$ lead to corrections of higher orders of smallness since the fluctuations of losses may be regarded as perturbations.

[7] A. Ahieser and V. Berestetskij: *Quantum Electrodynamics* (Moscow, 1955).

The solution of an inhomogeneous system corresponding to zero initial conditions has the following form:

$$(3.2)\quad \varrho(t) = \frac{c^2}{R}\exp\int_0^t\left[i\frac{c\sqrt{1-n}}{R} - \frac{\dot{E}_s}{2E_s} - \frac{n}{2(1-n)}\frac{\dot{W}_s}{E_s}\right]\mathrm{d}\zeta \int_0^t \frac{\mathcal{D}_b(x)\dot{w}(x)}{E_s(x)\mathcal{D}(x)}\,\mathrm{d}x +$$

$$+ \frac{c^2}{R}\exp\int_0^t\left[i\Omega - \frac{\dot{E}_s}{2E_s} + \frac{\dot{\Omega}}{2\Omega} - \frac{3-4n}{2(1-n)}\frac{\dot{W}_s}{E_s}\right]\mathrm{d}\zeta \int_0^t \frac{\mathcal{D}_c(x)\dot{w}(x)}{E_s(x)\mathcal{D}(x)}\,\mathrm{d}x + \text{c.c.}\,,$$

where $\mathcal{D}$ is the Wronskian of the homogeneous equation (2.8), and $\mathcal{D}_b$, $\mathcal{D}_c$ are the corresponding algebraic supplements.

Let us determine by means of (3.1) the value of the mean square amplitude ($\langle a_\varrho^2\rangle = 2\langle\varrho^2\rangle$). In order to do this we average over the photon spectrum and over the phase of betatron and synchrotron oscillations. Since the oscillation frequencies are widely differing, the averaging over each is performed independently, using the expression for the spectral density of radiation of the relativistic electron moving along a circumference (see ref. [5,8]).

$$(3.3)\qquad U(y)\,\mathrm{d}y = \frac{3\sqrt{3}}{4\pi}\frac{e^2c}{R^2}\gamma^4 y\,\mathrm{d}y\int_y^\infty K_{\frac{5}{3}}(x)\,\mathrm{d}x\,.$$

From (3.1), (3.2) (3.3) we obtain

$$(3.4)\qquad \langle a_{\varrho b}^2\rangle = \frac{55}{24\sqrt{3}}\frac{r_0\Lambda c}{R(1-n)^2\gamma}\int_0^t \exp\left[-\frac{n}{1-n}\int_x^t\frac{W_s}{E_s}\,\mathrm{d}\zeta\right]\gamma^6\,\mathrm{d}x\,,$$

$$(3.5)\qquad a_{\varrho c}^2 = \frac{55}{24\sqrt{3}}\frac{r_0\Lambda c V^{0.5}}{(1-n)^2R\gamma^{1.5}}\int_0^t \exp\left[-\frac{3-4n}{1-n}\int_x^t\frac{W_s}{E_s}\,\mathrm{d}\zeta\right]\gamma^{6.5}V^{-0.5}\,\mathrm{d}x\,,$$

where Λ is the Compton wave-length, and $r_0 = e^2/mc^2$ the « classical radius » of the electron.

Now consider briefly the qualitative physical picture of the discussed effect. At the moment of the emission of the photon ε_i, the radius of the closed orbit is decreased by the amount $\delta\varrho_c = -(R\varepsilon_i/(1-n)E_s)$ with the following change of ϱ_b by the amount $\delta\varrho_b = -\delta\varrho_c$. It is easily seen that during one half period these jumps lead to a decrease of the oscillation amplitude and during the other half period, to its increase. It is well known however that, owing to the statistical character of the process, the amplitude in absence of damp-

(8) D. IVANENKO and A. SOKOLOV: *Classical Theory of Fields* (Moscow, 1948).

ing increases, in the mean, as the square root of time (this is true for radial betatron as well as for radial synchrotron oscillations).

The process outlined above might be prolonged at infinitum. The taking into account of the familiar adiabatic damping ($\sim E_s^{-\frac{1}{2}}$ for ϱ_b and $\sim E_s^{\frac{1}{2}}\Omega^{\frac{1}{2}}$ for ϱ_c) does not substantially alter the picture, and the characteristic stochastic rising of oscillations $\sim \sqrt{t}$ is virtually maintained. This picture however is fundamentally changed if we take into account the radiation friction (2.13) and (2.14) mentioned above, which limits this rising to a damping time equal to $\approx ((1-n)/n)(E_s/W_s)$ for ϱ_b and $\approx ((1-n)/(3-4n))(E_s/W_s)$ for ϱ_c. This problem will be considered more in detail in the section devoted to arbitrary magnetic systems.

3·1. *Radiation fluctuations and vertical oscillations.* – The mechanism of excitation of z-oscillations by fluctuations differs from that of ϱ-oscillations outlined above. It is to be remembered that the photon is emitted in a direction deviating, in the mean, by a small angle from the direction of the electron velocity. At each act of photon emission a recoil pulse is imparted to the electron, which leads in the average to the excitation of oscillations similarly obeying the law $\sim \sqrt{t}$. In the case of ϱ-oscillations this effect leads to only small corrections to the expressions (3.4), (3.5), while in the case of z-oscillations it resumes the total effect of fluctuations.

It is to be noted, however, that radiation damping (2.10), limiting the duration of rising by the time of relaxation, impedes any significant increase of the amplitude a. Performing a series of computations according to the scheme outlined above and averaging over the photon spectrum, the angles of emission and the phases of vertical betatron oscillations, we finally obtain the following expression for a_z

$$\langle a_z^2 \rangle = \frac{13}{24\sqrt{3}} \frac{r_0 \Lambda c}{R n \gamma} \int_0^t \exp\left[-\int_x^t \frac{W_s}{E_s} \mathrm{d}\zeta \right] \gamma^4 \mathrm{d}x \,. \tag{3.6}$$

4. – Betatron oscillations in magnetic periodic systems in presence of radiation and methods for the damping of oscillations.

In the present Section we consider some peculiar features of radiation in magnetic periodic systems.

In the general case, when the curvature $\varkappa$ of the equilibrium orbit and the field index n are functions of the azimuth, the linearized equations of motion for ϱ, z, η are easily obtained (ref. (9)). As to their form these equations

(9) A. Kolomenskij: *Symposium CERN*, Geneva (1956); *Žu. Tehn. Fiz.*: **26**, 1969 (1956).

coincide with (2.4)–(2.6), but differ from the latter in that these parameters as well as $W_s(\varkappa)$ are functions of the generalized azimuth.

As usual the solution for ϱ will be sought in the form of superposition of a periodic solution corresponding to a closed orbit and betatron oscillations near this orbit

$$\varrho = \varrho_c + c \exp[i\mu m] f(\vartheta) + c^* \exp[-i\mu m] f^*(\vartheta) , \tag{4.1}$$

where $f(\vartheta)$ is the function of the azimuth depending on the magnetic structure and m the number of the successive elements of periodicity.

We shall limit our discussion to the more characteristic case of a system consisting of different sectors, with an axially-symmetric field in each of them (strong-focusing synchrotron, race-track).

From the results of Sect. 2, obtained for an axially-symmetric field, it follows that in each sector of a strong-focusing accelerator with $|n| \gg 1$ we will not have radiation damping as in (2.13), but radiation antidamping effect with approximately the same decrement. The physical meaning of this phenomenon has been elucidated in Sect. 2. A simple consideration shows that the same may be extended to accelerators in general. In the discussed periodic systems the amplitude of betatron oscillations varies according to the law

$$a_{\varrho b} \sim E_s^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\int_0^t (\alpha - 1)\, W_s \frac{\mathrm{d}x}{E_s}\right] , \tag{4.2}$$

where α is the momentum compaction factor.

According to evaluations in a strong-focusing accelerator the increase of amplitude by a factor e must take approximately $2 \cdot 10^{-2}/E$ (GeV) s, which gives to the effect of radiation excitation great practical importance.

Owing to this effect electron acceleration in strong-focusing accelerators up to energies of $5 \div 10$ GeV and even less becomes very difficult and even impossible unless some special provisions are taken.

4·1. *Methods of overcoming radiation antidamping.* – We have devised and studied several methods of overcoming the radiation antidamping.

A) One method of overcoming radiation antidamping of radial oscillations consists in the use of artificial strong coupling between radial and vertical oscillations. The latter in all systems are always subject to radiation damping according to the law (2.10), independently of n.

The working point must be located on the line of differential resonacen (ref. (5))

$$\mu_\varrho - \mu_z = \frac{2l\pi}{N}; \qquad l = 0, \pm 1 \ldots , \tag{4.3}$$

while the coupling is performed by rotating the sectors around the azimuthal direction or by introducing special magnetic lenses into the straight sections. Computation gives for radiation decrements of coupled radial and vertical oscillations the following expression

$$\xi_{b\varrho,z} = -\frac{1}{4}\alpha \mp \varkappa(2-\alpha)\frac{W_s}{E_s}, \tag{4.4}$$

where $\varkappa$ is the value characterizing the oscillations « bounding » which is equal to 1 in absence of coupling or at great enough distance from the resonance.

In the opposite case $\varkappa \to 0$ and we obtain for the decrements the expression

$$\xi_{b\varrho,z} \approx -\frac{1}{4}\alpha\frac{W_s}{E_s}. \tag{4.5}$$

It will be seen that practically neither ϱ_b nor z-oscillations experience any damping or antidamping, while at the same time the amplitude of vertical oscillations equals that of radial oscillations which rapidly rise in absence of damping owing to fluctuations.

B) A damping of radial oscillations might be achieved also by creating the accelerating voltage amplitude dependence on the radius $V_0 = V_0(r)$. This method is not directly related to radiation, but may be used with good results in powerful electronic accelerators in which the accelerating voltage has to be very high owing to radiation.

It is physically obvious that this dependance will affect both betatron and synchrotron oscillations. Successive computations give the following results:

a) Synchrotron oscillations are subject to additional damping with the decrement

$$\xi_c = \frac{\langle \psi n_v\rangle}{2}\frac{\varkappa e V_0 \cos\varphi_s c}{E_s L_s} \simeq \frac{1}{2}\frac{\varkappa W_s}{E_s}\langle \psi n_v\rangle, \qquad n_v = -\frac{R}{V_0}\frac{\partial V_0}{\partial r}, \tag{4.6}$$

where for definition of ψ see (5.1) and L_s is the equilibrium orbit's length. Since in strong-focusing accelerators the value of α is small ($\alpha \approx 5/n$) the effect of damping becomes felt only at great n_v.

b) The damping of betatron oscillations, owing to the dependence $V_0(r)$ is characterized by the decrement

$$\xi_{b\varrho} = -\frac{1}{2}\langle \psi n_v\rangle \frac{\varkappa e V_0 \cos\varphi_s c}{E_s L_s} \simeq -\frac{1}{2}\langle \psi n_v\rangle \frac{\varkappa W_s}{E_s}. \tag{4.7}$$

As seen from (4.6)–(4.7) the damping of synchrotron oscillations is followed by equivalent antidamping of betatron oscillations and vice versa.

Nevertheless this last method may prove useful for the damping of synchrotron oscillations at the expense of betatron oscillations, since the former may prove more dangerous and vice versa.

A better result may be achieved in a strong-focusing accelerator by using a peculiar damping system by means of a variable azimuth voltage gradient, for instance according to following scheme

$$\begin{array}{ccc} \dfrac{\partial V_0}{\partial r} < 0 & & \dfrac{\partial V_0}{\partial r} > 0 \\ \boxed{n<0}\,\boxed{n>0} & \downarrow & \boxed{n>0}\,\boxed{n<0} \\ & & \uparrow \\ & & \boxed{n<0}\,\boxed{n>0} \end{array} .$$

As shown by computations, this may lead to the damping of betatron oscillations, while the synchrotron oscillations remain practically unaffected. An alternative, less convenient, method consists in making the sectors $n > 0$ and $n < 0$ of different length.

5. – Effect of radiation fluctuations on betatron and synchrotron oscillations in magnetic periodic structures.

The effect of radiation fluctuations for this case can be taken into account in the same manner as for the case of an axial-symmetrical accelerator (see Sect. 3) by solving the corresponding Hill's inhomogeneous equations and averaging. In order to make this picture physically more clear, we give here another deduction of the expressions for the mean square values $\langle a^2_{\varrho b}\rangle$ and $\langle a^2_{\varrho c}\rangle$, $\langle a^2_{\eta c}\rangle$ separately.

5·1. *Betatron oscillations.* – Suppose that on a certain azimuth the electron emits a photon with energy $\varepsilon = \hbar\omega$. This gives rise to a jump-like change of the localization of the closed orbit and its slope in the said point, with a simultaneous change of the analogous characteristics of the betatron oscillations

$$\left\{ \begin{array}{l} \delta\varrho_c = -\,\delta\varrho_b = -\,\dfrac{\varepsilon}{E_s}\,\psi(\vartheta)\,, \\ \delta\varrho'_c = -\,\delta\varrho'_b = -\,\dfrac{\varepsilon}{E_s}\,\psi'(\vartheta)\,, \end{array} \right. \tag{5.1}$$

where $\psi(\vartheta)$ is the known periodic function depending on the type of the magnetic system and having the same period (see ref. [9]).

Using (5.1) and the familiar normalization of the Wronskian $W(ff^*) = -2i$ we obtain

$$\langle \delta(a_{\varrho b})^2 \rangle = |f|^2_{\max} [\langle |f'|^2 (\delta\varrho_b)^2 \rangle + \langle |f|^2 (\delta\varrho_b')^2 \rangle] . \tag{5.2}$$

The total change of the value $\langle a^2_{\varrho b} \rangle$ consists of two parts: the systematic changes of (4.2) and of the stochastic part (5.2).

Taking this into account and using (5.2) we obtain, according to a scheme similar to that used in Sect. 3, an expression for the determination of $\langle a^2_{\varrho b} \rangle$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle a^2_{\varrho b} \rangle = \frac{55}{24\sqrt{3}} r_0 c \Lambda F \gamma^5 - \frac{\langle n W_s/(1-n) \rangle + \dot{E}_s}{E_s} \langle a^2_{\varrho b} \rangle , \tag{5.3}$$

where F is a factor depending on the parameters and configuration of the system and equal to

$$F = |f|^2_{\max} [\langle |f'|^2 \psi^2 \varkappa^3 \rangle + \langle |f|^2 \psi'^2 \varkappa^3 \rangle] , \tag{5.4}$$

where $\varkappa$ is the curvature of the equilibrium orbit. The averaging in (5.4) is performed over the element of periodicity, including the sections free of field, to which corresponds $\varkappa = 0$.

In the case of an axially-symmetric field $F \equiv 1/R(1-n)^2$ and we obtain again the formulas of Sect. 2, 3.

By integrating (5.3) we obtain the expression:

$$\langle a^2_{\varrho b} \rangle = \langle a^2_{\varrho b} \rangle_0 \exp\left[-\int_0^t ((\alpha - 1) W_s + E_s) \frac{\mathrm{d}x}{E_s} \right] + \tag{5.5}$$

$$+ \frac{55}{24\sqrt{3}} r_0 \Lambda c F \int_0^t \gamma^5 \exp\left[-\int_x^t ((\alpha - 1) W_s + E_s) \right) \frac{\mathrm{d}\zeta}{E_s} \Big] \mathrm{d}x ,$$

which is a generalization of (3.4).

If, owing to the small value of energy or of the acceleration time, the radiation friction has no means of manifesting itself, (5.5) leads to:

$$\langle a^2_{\varrho b} \rangle_{\mathrm{in}} = \langle a^2_{\varrho b} \rangle_0 \frac{E_{0s}}{E_s} + \frac{55}{24\sqrt{3}} \frac{r_0 \Lambda c F}{\gamma} \int_0^t \gamma^6 \mathrm{d}x . \tag{5.6}$$

The first quantity in (5.6) corresponds to the « classical » part of betatron oscillations, while the second expresses its « fluctuation » part. Their behaviour,

in general, essentially dependent on the sign and magnitude of the parameter

$$\xi_{qb} = -(\alpha - 1)\frac{W_s}{E_i}.$$

If $\xi_{qb} > 0$ both parts of (5.3) are subject to the antidamping discussed above, the nature of which, as was demonstrated, is purely classical.

If a damping system is used we obtain from (5.5) an established value of the amplitude. For example

$$(5.7)\quad \langle a_{qb}^2 \rangle_{\text{est}} = \frac{55\sqrt{3}}{48}\,\frac{\Lambda F}{\varkappa^2}\,\gamma^2.$$

We give below a table showing the values of F as a function of the selection of the working point in a strong-focusing magnet.

For the sake of simplicity consider the case when the gradient in the sectors of different types has the same absolute value, and the sectors and straight sections are of the same size,

$$|n_1| = n_2 = n \gg 1; \qquad \nu_1 = \nu_2 = \nu;$$

$$l_1 = l_2 = l; \qquad F = \frac{F_1}{n^2 R}; \qquad \varkappa = \frac{1}{R}.$$

In case of a weak-focusing accelerator with straight section (race-track) the factor F differs from $1/R(1-n)^2$ by a few percents only. Fig. 1 shows a typical dependence of the amplitude of radial betatron oscillations on the magnitude of $\xi = t/T_{ac}$, where T_{ac} is the time of acceleration; the approximating curves I and II correspond to (5.6) and (5.7).

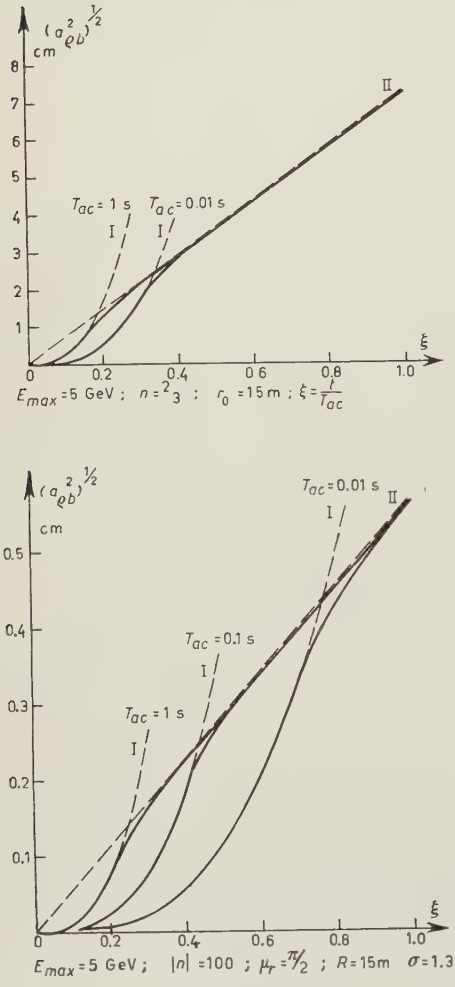

Fig. 1.

Note that for vertical oscillation, as a result of a similar computation, we obtain:

$$\langle a_z^2 \rangle_{\text{est}} = \frac{13\sqrt{3}}{48}\,\Lambda\,\frac{|f|^2_{\max}}{\langle \varkappa^2 \rangle}\,\langle \varkappa |f|^2 \rangle.$$

The established values of their amplitude are independent of energy and for all practically interesting cases are negligibly small.

TABLE OF THE VALUES OF FACTOR F_1

μ	$l/Rv=0$	$l/Rv=0,3$	$l/Rv=0,5$
0.78	850	430	330
1.30	530	220	180
1.65	510	210	150
1.94	620	310	110

5·2. *Synchrotron oscillations.* – Consider separately the slow synchrotron oscillation of radius ϱ_c, phase η_c and energy ΔE_c. For simplicity we shall discuss only the case when the curvature of the orbit in all magnetic sectors has the same value $\varkappa = 1/R$. According to phase determination

$$\dot{\eta}_c = q(\omega_{rs} - \omega_r) , \tag{5.8}$$

where ω_r is the frequency of revolution, we obtain the following useful relation:

$$\dot{\eta}_c = \frac{qc}{R^2\sigma^2} \langle \psi \rangle_H \frac{\Delta E_c}{E_s} , \tag{5.9}$$

where $\langle \psi \rangle_H$ denotes an averaging over the sector with magnetic field. The radiation power of the particle $W(r_c)$ moving with energy $E_c = E_s + \Delta E_c$ in a magnetic sector along a closed orbit of radius $r_c = R + \varrho_c(\vartheta)$, according to (1.1), has the form:

$$W(r_c) = W_s \left\{ 1 + 2 \left[1 - \frac{n\psi(\vartheta)}{R} \right] \frac{\Delta E}{E_s} \right\} , \tag{5.10}$$

where W_s denotes the power of the radiation losses during the motion along the circular section of the equilibrium orbit with equilibrium energy $W_s = W(R, E_s)$. Now using the law of conservation of energy for the non-equilibrium particle and taking into account the presence of radiation fluctuations, the phase equation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \Delta E_c = \left(\frac{\mathrm{d}}{\mathrm{d}t} \Delta E_c \right)_{\mathrm{ac}} - \left(\frac{\mathrm{d}}{\mathrm{d}t} \Delta E_c \right)_{\mathrm{rad}} - \frac{1}{\sigma} \sum_j \varepsilon_j (a_j - a_j) . \tag{5.11}$$

The quantities $((\mathrm{d}/\mathrm{d}t)\Delta E_c)_{\mathrm{ac}}$ and $((\mathrm{d}/\mathrm{d}t)\Delta E_c)_{\mathrm{rad}}$ denote the variation of the value of ΔE_c per unit of time due to the accelerating field, and the radiation losses respectively. The last quantity in the right hand side describes the radiation

fluctuations characterized by the following values: ε_j the energy of the emitted photons of the j-th harmonic, and a_j the number of photons emitted per unit of time by an electron moving in a magnetic field; factor $\sigma = 1 - (l/Rv)$ takes into account the presence of sections free of field and radiation.

Using the expressions derived above and (5.11) we obtain the linearized phase equation for slow synchrotron oscillations

$$\ddot{\eta} + \left(\frac{\dot{E}_s}{E_s} + \frac{Q}{\sigma}\frac{W_s}{E_s}\right)\dot{\eta} + \Omega^2\eta = \frac{qc\langle\psi\rangle_H}{\sigma^3 R^2 E_s}\sum_j \varepsilon_j(a_j - \bar{a}_j)\,, \tag{5.12}$$

where Ω is the frequency of phase oscillations

$$\Omega^2 = \frac{qc^2 eV\langle\psi\rangle_H}{2\pi\sigma^2 R^3 E_s}\,, \tag{5.13}$$

and Q denotes the parameter

$$Q = 2 + \frac{\langle\psi(1-2n)\rangle_H}{R}\,. \tag{5.14}$$

From this general equation may be derived the result obtained by SANDS for the case of a race-track (ref. [3]) by correcting an error in [3], i.e. by multiplying the right-hand part of the equation by a factor $1/\sigma$ which takes into account the decrease of fluctuation effects due to the presence of straight sections.

Solving (5.12) we obtain the mean square value $\langle a^2_{\eta c}\rangle$

$$\langle a^2_{\eta c}\rangle = \frac{55\pi}{12\sqrt{3}}\,\frac{qe\Lambda c\langle\psi\rangle_H V^{-1.5}\gamma^{0.5}}{R^4\sigma^3}\int_0^t \exp\left[-\int_x^t \frac{QW_s}{\sigma E_s}\,\mathrm{d}\xi\right]\gamma^{5.5}V^{0.5}\,\mathrm{d}x\,. \tag{5.15}$$

Hence at $\int_0^t (QW_s/\sigma E_s)\,\mathrm{d}\xi \gg 1$ the established value may be easily obtained

$$\langle a^2_{\eta c}\rangle_{\mathrm{est}} = \frac{55\sqrt{3}}{32}\,\frac{q\Lambda}{r_0\sigma^2}\,\frac{\langle\psi\rangle_H}{RQ\gamma}\operatorname{ctg}\varphi_s\,. \tag{5.16}$$

In case of non-established oscillations, when radiation damping is still weak, which corresponds to $\int_0^t (QW_s/\sigma E_s)\,\mathrm{d}\xi \ll 1$, we obtain from (5.15) the following expression

$$\langle a^2_{\eta c}\rangle_{\mathrm{in}} = \frac{55}{24\sqrt{3}}\,\frac{r_0\Lambda c^2\langle\psi\rangle_H q\operatorname{ctg}\varphi_s}{R^5\sigma^4\dot{\gamma}}\left(\frac{\gamma}{V}\right)^{0.5}\int_0^t \gamma^{5.5}V^{0.5}\,\mathrm{d}x\,. \tag{5.17}$$

For the case of weak-focusing

$$Q = \frac{3-4n}{1-n}\,. \tag{5.18}$$

For strong-focusing, according to (5.14), we have for the centre of the stable region

$$Q \approx 4 .$$

The total behaviour of the function $\langle a_{\eta c}^2 \rangle^{\frac{1}{2}}(\xi)$ is shown in Fig. 2 for both weak and strong-focusing.

It must be noted that the amplitude of fluctuation excited phase oscillations is rather large and slowly decreases in the region of high energies (as $\gamma^{-\frac{1}{2}}$). Therefore the azimuthal size of the accelerated particle beam is great enough to practically inhibit all coherent radiation ([3,10]). As seen from (5.15), (5.17) (see also Fig. 2), in the beginning the value $\langle a_{\eta c}^2 \rangle^{\frac{1}{2}}$ increases. The maximum value, however, is not necessarily reached at the end of acceleration but might be reached long before that, depending on the acceleration time and other parameters.

In the latter case $\langle a_{\eta c}^2 \rangle^{\frac{1}{2}}$ passes through a maximum with a further decrease according to (5.16). If the maximum value of $\langle a_{\eta c}^2 \rangle^{\frac{1}{2}}$ is comparable with the size of the separatrix, it will (unless some special measures are taken) lead to the loss of a fraction of accelerated particles.

We give below the formulae for the mean square of radial synchrotron oscillations. Note, that the maximum deviation ϱ_c which plays the role of the amplitude $a_{\varrho c}$, is equal to

$$a_{\varrho c} = \frac{\psi_{\max}}{E_s} (\Delta E)_{\max} , \tag{5.20}$$

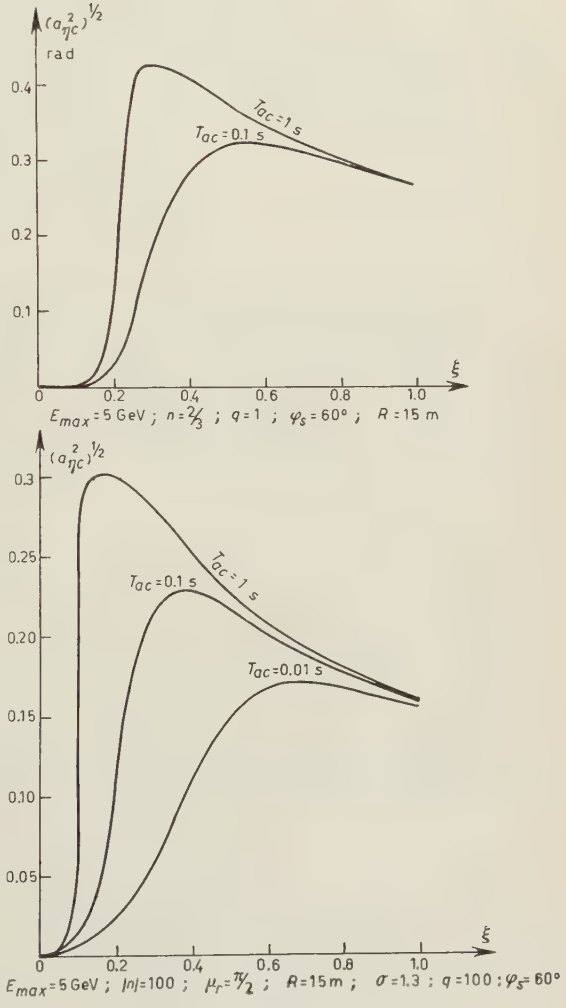

Fig. 2.

where $\psi_{\max}$ is the maximum value of $\psi(\vartheta)$. Using the relation between i_c and ϱ_c (see (5.9)) and the expression

([10]) J. Nodvick and D. Saxon: *Phys. Rev.*, **96**, 180 (1954).

(5.13), from (5.16), (5.17) and taking into account adiabatic damping, we obtain

$$\langle a_{\varrho c}^2 \rangle_{\text{est}} = \frac{55\sqrt{3}}{48} \frac{\Lambda r_0 \psi_{\max}^2 \gamma^2}{Q\sigma}, \tag{5.21}$$

$$\langle a_{\varrho c}^2 \rangle_{\text{in}} = \frac{55}{24\sqrt{3}} \frac{r_0 \Lambda c \psi_{\max}^2 V^{0.5}}{R^3 \sigma^2 \gamma^{1.5}} \int_0^t \gamma^{6.5} V^{-0.5} \, \mathrm{d}x . \tag{5.22}$$

The absolute value of the excited radial synchrotron oscillations is very small. Thus from (5.21) which gives the highest possible value at a given fixed E_s and R we obtain the approximate evaluation

$$\langle a_{\varrho c}^2 \rangle^{\frac{1}{2}} \simeq \frac{E^{\frac{3}{2}}\,(\text{GeV})}{|n|}\ \text{cm}, \tag{5.23}$$

which at $|n| \gtrsim 100$, even for $E = 10$ GeV, gives only a few mm.

In resuming it must be noted that the outlined scheme may be extended with small modifications to the computation of the influence of a variety of perturbing effects, involving statistical and fluctuation phenomena (bremsstrahlung on the residual gaz nuclei, Coulomb interactions) as well as to the computation of corrections due to discrete feeding of energy into the accelerating gaps (« slit » oscillations).

SUPPLEMENTO AL VOLUME VII, SERIE X
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On the Modification of Feynman's « Integral-over-All-Paths » Method.

J. ŁOPUSZAŃSKI

Institute of Theoretical Physics of the Polish Academy of Sciences
Wrocław, Poland

(ricevuto il 28 Giugno 1957)

CONTENTS. — **1.** Introduction. – **2.** Rigorous solution of equations of diffusion type. – **3.** On the Feynman's « integral-over-all-paths ». – **4.** Transition to continuous t-variable. – **5.** The usual perturbation calculus.

1. – Introduction.

We begin with the rigorous solution of some equations of diffusion type. Further we investigate Feynman's « integral over-all-paths » and use our previous results to modify the computation technique of this method. One may say roughly that the relation between our method of summation and that used by FEYNMAN [1] is similar to the relation between the Lebesgueian and Riemannian integral. Some considerations, especially of combinatorial and probabilistic character, in the second part of the note will be only outlined. More exhaustive treatment will be published soon [2].

2. – Rigorous solution of equations of diffusion type.

2·1. – Let us call an equation of the form

$$\frac{\partial \psi(j, j_0; t)}{\partial t} = \lim_{\Delta t \to 0} \left[\int z(j \mid j') \, \mathrm{d}j' \, \psi(j', j_0; t - \Delta t) - \psi(j, j_0; t - \Delta t) \right] \Big|_{-t}^{1} . \tag{1}$$

[1] R. P. FEYNMAN: *Revs. Mod. Phys.*, **20**, 367 (1948).
[2] J. ŁOPUSZŁAŃSKI: *Acta Phys. Pol.* (in press).

an equation of the diffusion type. The unknown function is $\psi(j, j_0, t)$ and and $z(i \mid k)$ are given functions; they are elements of the transition matrix for a time interval Δt; the initial condition for $t = 0$ is

$$\psi(j, j_0; t=0) = \begin{cases} \delta(j - j_0) & \text{if } j \text{ is continuous} \\ \delta_{jj_0} & \text{if } j \text{ is discrete.} \end{cases} \tag{2}$$

To this type of equations belong e.g. the Schrödinger equation, equations of stochastic processes as well as equation describing the sum-over-states (in these examples, except the last, t has the meaning of time). If the j spectrum of states is not continuous we replace the integral on the right-hand side of (1) by a sum (or Stieltjes integral).

For the sake of simplicity we confine ourselves in our considerations to the case when the j-spectrum is discrete and the number of levels is bounded, say l. We assume also that the variable t is discrete. Finally we will discuss the case of continuous t. Thus we consider the equation

$$\psi(j, j_0; n) = \sum_{j'=1}^{l} z(j \mid j') \psi(j', j_0; n-1) , \tag{1a}$$

where $t = n\Delta t$ and $j, j_0 = 1, ..., l$; $n = 1. 2, ...$

2·2. – To solve (1a) we define the generating function for $\psi(j, j_0; n)$

$$\Phi(j, j_0; \lambda) = \sum_{n=0}^{\infty} \lambda^n \psi(j, j_0; n) \tag{3}$$

convergent uniformly in the neighbourhood of $\lambda = 0$ (*). It follows from (1a) that Φ satisfies the equation

$$\Phi(j, j_0) = \delta_{jj_0} + \lambda \sum_{y} z(j \mid y) \Phi(y, j_0) . \tag{4}$$

Equation (4) yields the solution

$$\Phi(j, j_0; \lambda) = \frac{J_{jj_0}}{J} ,$$

(*) If we want to use a continuous t-parameter, we must replace the power series on the right-hand side of (3) by a Laplace transform of $\psi(t)$.

where

$$J(\lambda) = \begin{vmatrix} -\lambda z(1|1)+1, & -\lambda z(2/1), & \dots & -\lambda z(l|1) \\ -\lambda z(1|2), & -\lambda z(2|2)+1, & \dots & -\lambda z(l|2) \\ \dots & \dots & \dots & \dots \\ -\lambda z(1|l), & -\lambda z(2|l), & \dots & -\lambda z(l|l)+1 \end{vmatrix}$$

is the determinant of the « stochastic » matrix (we write inverted commas because $z(i|k)$ must not be probabilities) and J_{xy} is the minor of $J(\lambda)$. We put this solution into the form

$$\Phi(j, j_0; \lambda) = \frac{\partial}{\partial z(j_0|j)} \Omega(\lambda) , \tag{5a}$$

with

$$\Omega(\lambda) = -\frac{1}{\lambda} \ln J(\lambda) . \tag{5b}$$

It is obvious from (3) and (5*a*) that, if we develop Ω in a power series in respect to λ

$$\Omega(\lambda) = \sum_{n=0}^{\infty} \lambda^n \omega(n) \tag{6a}$$

we get

$$\psi(j, j_0; n) = \frac{\partial \omega(n)}{\partial z(j_0|j)} . \tag{6b}$$

The solution (5) or (6) has an interesting feature: each solution of a problem concerning a process of diffusion (Markoff processes, Schrödinger equation, etc). may be written as a partial derivative in respect to a transition matrix element, say $z(x_0|x)$, of a « potential function » $\omega(t)$. This function is symmetrical in $z(i|k)$. For example we are able to write all elements of the S matrix as « gradient components » of only one such « potential function » in the $z(i|k)$ - space.

2·3. – We prove now the relation

$$\frac{1}{J} = 1 + \sum_{n=0}^{\infty} \lambda^{n+1} \left\{ \sum_{\mu} \prod_{i=1}^{l} \prod_{k=1}^{l} z(i|k)^{\mu_{ik}} \frac{\prod_k \mu_k!}{\prod_i \prod_k \mu_{ik}!} \right\} , \tag{7}$$

where the summation is submitted to following restrictions

$$\sum_{j=1}^{l} \mu_{ij} = \sum_{j=1}^{l} \mu_{ji} = \mu_i , \qquad \sum_{i=1}^{l} \sum_{j=1}^{l} \mu_{ij} = n+1 . \tag{7a}$$

Let us write the right-hand side of (7) in the form

$$1+\sum_{n=0}^{\infty}\lambda^{n+1}\left\{\sum_{\mu_1}\dots\sum_{\mu_l}\left(\frac{1}{2\pi i}\right)^l\oint_{C_1}\dots\oint_{C_l}\prod_{k=1}^{l}\left[\sum_{i=1}^{l}z(i\,|\,k)\frac{\zeta_i}{\zeta_k}\right]^{\mu_k}\prod_{k=1}^{l}\frac{\mathrm{d}\zeta_k}{\zeta_k}\right\}, \tag{8}$$

where we sum over μ_i, for which

$$\sum_{i=1}^{l}\mu_i=n+1 \tag{8a}$$

holds, and the integrals are taken along closed paths C_i in the ζ_i complex-planes, enclosing the origins of co-ordinates resp. Now we are able to perform in (8) the summation over n. We have

$$1+\underbrace{\sum_{\mu_1}\dots\sum_{\mu_l}}_{\sum_{i=1}^{l}\mu_i=1}\left(\frac{1}{2\pi i}\right)^l\oint_{C_1}\dots\oint_{C_l}\prod_{k=1}^{l}\left[\sum_{i=1}^{l}z(i\,|\,k)\frac{\zeta_i\lambda}{\zeta_k}\right]^{\mu_k}\frac{\mathrm{d}\zeta_k}{\zeta_k}- \tag{9}$$

$$-\left(\frac{1}{2\pi i}\right)^l\oint_{C_1}\dots\oint_{C_l}\prod_{k=1}^{l}\frac{\mathrm{d}\zeta_k}{\zeta_k-\lambda\sum_{i=1}^{l}z(i\,|\,k)\zeta_i}.$$

We introduce new variables

$$\vartheta_k=\zeta_k-\lambda\sum_{i=1}^{l}z(i\,|\,k)\zeta_i\,;$$

then we have

$$\prod_{k=1}^{l}\mathrm{d}\zeta_k=\frac{1}{J}\prod_{k=1}^{l}\mathrm{d}\vartheta_k\,;$$

consequently

$$\left(\frac{1}{2\pi i}\right)^l\oint_{C_1}\dots\oint_{C_l}\prod_{k=1}^{l}\frac{\mathrm{d}\zeta_k}{\zeta_k-\lambda\sum_{i}z(i\,|\,k)\zeta_i}-\frac{1}{J}\left(\frac{1}{2\pi i}\right)^l\oint_{C_1'}\dots\oint_{C_l'}\prod_{u=1}^{l}\frac{\mathrm{d}\vartheta_k}{\vartheta_k}=\frac{1}{J}, \tag{10}$$

where the paths C_i' are similar to C_i, but in ϑ_k-planes.

2·4. – The function

$$\frac{1}{J}-1=\lambda K(\lambda) \tag{11}$$

tends to zero together with λ. Consequently

$$\Omega(\lambda)=\frac{1}{\lambda}\ln\left(1+\lambda K(\lambda)\right),$$

may be developed in the neighbourhood of $\lambda = 0$ in a power series in respect to λ. In this way we get

$$\omega(n) = \sum_{m=1}^{n+1} (-1)^{n+1} \frac{1}{m} \underbrace{\sum_{\mu_1^1} \dots \sum_{\mu_1 n} \dots \sum_{\mu_l^1} \dots \sum_{\mu_l^m}}_{\sum_{s=1}^{m} \sum_{j=1}^{l} \mu_j^s = n+1} \prod_{i=1}^{m} a_{\mu_1^i \dots \mu_l^i} \tag{12}$$

where

$$\sum_{\mu} \prod_{k=1}^{l} \prod_{i=1}^{l} z(i|k)^{\mu_{ik}} \frac{\mu_k!}{\mu_{ik}!} = \sum_{\mu_1} \dots \sum_{\mu_l} a_{\mu_1 \dots \mu_l} , \tag{12a}$$

the summation on the left-hand side is submitted to condition (7*a*).

3. – On the Feynman's « integral-over-all-paths ».

3·1. – FEYNMAN (reference [1]) proposed a new method to solve the Schrödinger equation by means of so called « integral-over-all-paths ».

In this section we will present a modification of this method; we have not, however, in view to modify the main idea of this method but only the mathematical formalism.

Let us illustrate firstly the method of Feynman and further our approach on the example of a Schrödinger equation in one space dimension (*).

We ask for the solution

$$\psi(x, x_0; t) = \left\langle x \left| \exp\left[-\frac{i}{\hbar} H t\right] \right| x_0 \right\rangle ,$$

of equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi \equiv H\psi , \tag{13}$$

with the initial condition

$$\psi(x, x_0; t = 0) = \delta(x - x_0) . \tag{14}$$

This solution may be written by means of the Feynman's method

$$\psi(x, x_0; t) = \lim_{\substack{\Delta t \to 0 \\ n \Delta t = t}} \underbrace{\int \dots \int}_{n-1} \mathrm{d}x_1 \dots \mathrm{d}x_{n-1} \prod_{i=1}^{n} \left(\frac{m}{2\pi\hbar \Delta t i}\right)^{\frac{1}{2}} \exp\left[\frac{i}{\hbar} L(x_i, x_{i-1}) \Delta t\right] , \tag{15a}$$

(*) We perform our computations in one space-dimension, the generalization to more than one dimension seems to be easily done and is outlined in the last but one section of this note.

where

$$L(x_i, x_{i-1}) = \frac{1}{2} m \left(\frac{x_i - x_{1-i}}{\Delta t} \right)^2 - V(x_i) \tag{15b}$$

and $x_n = x$. In (15a)

$$\left(\frac{m}{2\pi\hbar \Delta t i} \right)^{\frac{1}{2}} \exp\left[\frac{i}{\hbar} L(x_i, x_{i-1}) \Delta t \right] = \left\langle x_i \left| \exp\left[-\frac{i}{\hbar} H \Delta t \right] \right| x_{i-1} \right\rangle , \tag{16}$$

is the transition probability amplitude from state x_{i-1} to state x_i in thetime interval Δt. If we denote, as in Sect. 2, the transition amplitude from state k to j in the time interval Δt by $z(j\,|\,k)$, the solution (15) may be, of course, written

$$\psi(x, x_0; t) = \lim_{\substack{\Delta t \to 0 \\ n\Delta t = t}} \underbrace{\int \dots \int}_{n=1} \mathrm{d}x_1 \dots \mathrm{d}x_{n-1} \prod_{i=1}^{n} z(x_i\,|\,x_{i-1}) . \tag{15c}$$

We conclude from (15c) and Sect. 2, that Feynman's method may be applied to each equation of diffusion type; the solution of such equation has the form (15c) and may be immediately written down, if we only know the transition matrix $z(j\,|\,k)$.

So far Feynman's idea.

In formulae (15a) the $(n-1)$-fold integration appears; this summation procedure was already modified by some authors (see e.g. [3]). We propose next another way of computation: we replace the $(n-1)$-fold integration by sums over groups of « similar » paths. What will be understood by « similar paths » will be clarified further.

3·2. Let us demonstrate our attempt on the simplest example, when x has discrete levels and the number of levels is bounded, as in Sect. 2. Although this case is meaningless for physical practice (except perhaps the case of spin waves in a crystal), nevertheless this example is suitable to present clearly the main idea of our speculations, without troublesome computations. This case was already investigated and rigorously solved in Sect. 2. As in Sect. 2, we assume also the time to be a discrete variable, the shortest interval being Δt.

Thus the solution (15c) becomes

$$\sum_{x_1=1}^{l} \dots \sum_{x_{n-1}=1}^{l} \prod_{i=1}^{n} z(x_i\,|\,x_{i-1}) . \tag{15d}$$

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[3] B. Davison: *Proc. Roy. Soc.*, A **225**, 252 (1954).

This solution consists of a sum over « chains » of length n, which all start in x^0 and end in x. Such a « chain » may be also expressed by

$$\prod_{i=1}^{n} z(x_i \,|\, x_{i-1}) = \prod_{j=1}^{l} \prod_{k=1}^{l} z(j \,|\, k)^{m_{jk}} , \tag{17}$$

where m_{jk} denotes how many times the term $z(j\,|\,k)$ appears in the concerned « chain ». We have, of course,

$$\sum_{j=1}^{l} \sum_{k=1}^{l} m_{jk} = n . \tag{18a}$$

Each « chain » $\prod_{i=1}^{n} z(x_i \,|\, x_{i-1})$ may be written in a unique manner in the form (17) on the right-hand side, but not *vice versa*; the number of « chains » which may be characterized by the same ensemble of exponents m_{jk} and which begin in x_0 and end in x we denote by

$$\langle x \,|\, M \,|\, x_0 \rangle , \tag{19}$$

we quote by $\boldsymbol{M}$ the dependence of (19) from all matrix elements m_{jk}

$$\boldsymbol{M} = \begin{matrix} m_{11}, & m_{12}, & \dots & m_{1l} \\ m_{21}, & m_{22}, & \dots & m_{2l} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ m_{l1}, & m_{l2}, & \dots & m_{ll} \end{matrix} . \tag{19a}$$

We mention that if $\boldsymbol{M}$ is given the length n may be immediately fixed by means of (18*a*).

The necessary condition to form at least one « chain », if $\boldsymbol{M}$ is fixed, is

$$\sum_{j} m_{jk} = \sum_{j} m_{kj} \qquad \text{for } k \neq x, x_0 \tag{18b}$$

and

$$\sum_{j} m_{jx} + 1 = \sum_{j} m_{xj} , \tag{18c}$$

$$\sum_{j} m_{j,x_0} - 1 = \sum_{j} m_{x_0,j} . \tag{18d}$$

Relations (18*b-d*) mean simply that in a « chain » the number of transitions from a fixed state, say k, must be equal to the number of transitions towards this fixed state, excepted x and x_0. If we exclude from considerations the

case when $\boldsymbol{M}$ is only diagonal, then (18) isalso a sufficient condition to build a « chain » of length n. It follows from (18) at once that

$$\sum_j \sum_k (j - k) m_{jk} = x - x_0 \,, \tag{20}$$

consequently the final state x is uniquely determined by $\boldsymbol{M}$ and x_0. Thus we call some « chains » or paths « similar », if they are characterized by the same $\boldsymbol{M}$ and x_0.

Using notation (17) and (19) the solution (15*d*) may be written

$$\psi(x, x_0; n) = \sum_{\boldsymbol{M}} \prod_{j=1}^{l} \prod_{k=1}^{l} z(j|k)^{m_{jk}} \langle x | \boldsymbol{M} | x_0 \rangle \,, \tag{15e}$$

the summation is extended only to such $\boldsymbol{M}$'s for which (18*a*) holds. Of course, the $\langle x | \boldsymbol{M} | x_0 \rangle$'s, for which (18*b-d*) do not hold, vanish.

The expression on the right-hand side of (15*e*) has the form of a multiple power series of the variables $z(j|k)$; thus the $\langle x | \boldsymbol{M} | x_0 \rangle$'s play the rôle of power coefficients.

It should be emphasized that the nature of the physical problem is entirely involved in $z(j|k)$'s (e.g. the mode of interaction of the particles of the system is described by their potential energy appearing only in the exponent of the transition probability amplitude, in a similar manner as already shown in this section for the one dimensional Schrödinger equation in x-representation). On the other hand the expressions $\langle x | \boldsymbol{M} | x_0 \rangle$'s do not depend at all on the physical nature of the problem; to find them we must only employ geometrical or combinatorial considerations.

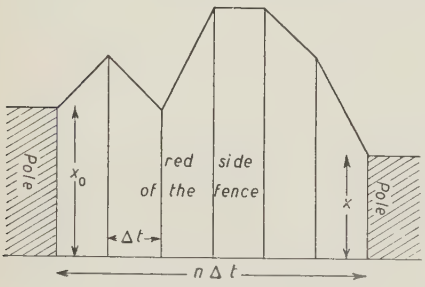

Fig. 1.

The problem of evaluation of $\langle x | \boldsymbol{M} | x_0 \rangle$ may be called « the problem of construction of fences » (see Fig. 1). Indeed: let us imagine, we have got a transport of planks of Δt cm width. Each of the planks is painted on the one side blue and on the other red. From among these planks m_{jk} is cut-off slantwise so that, looking at the red side, the left border has the h-ight k cm and the right — j cm (see Fig. 2). The number of all planks is n. Imagine further, that we have two poles in the distance $n\Delta t$ cm one from another. One has the height x_0 cm, the other — x cm.

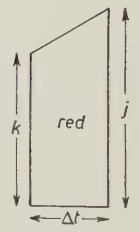

Fig. 2.

We ask in how many ways one may build a fence between ends so that the upper sides of the planks form a continuous, broken line with these poles, of the height of the pole tops resp. (see Fig. 1). Each side of the

fence is uniform in colour; if we look of the red side we should have the pole of x_0 height on our left-hand side.

3·3. – We show now how we can evaluate approximately the expression $\langle x|\boldsymbol{M}|x_0\rangle$. We use combinatorial considerations as well as make extensive use of the law of large numbers. We omit here, however, these considerations, because they are rather long, although they are interesting and instructive. These heuristic considerations will be published soon in extenso in a longer article (see reference [2]). We quote now only the final result, giving a good approximation for large n,

$$\langle x|\boldsymbol{M}|x_0\rangle \cong \frac{\mu_{x_0 x}}{n+1} \frac{\prod_j \mu_j!}{\prod_k \prod_i \mu_{kj}!} \tag{21}$$

for μ_{jk} sqtisfying (7a), and the generating function

$$\tilde{\Phi}(x, x_0; \lambda) = \frac{\partial}{\partial z(x_0|x)} \tilde{\Omega}(\lambda) , \tag{22a}$$

with

$$\tilde{\Omega}(\lambda) = \frac{1}{\lambda}\int \frac{\mathrm{d}\lambda}{\lambda}\left(\frac{1}{J}-1\right) . \tag{22b}$$

We show, that the functions $\tilde{\Phi}$ satisfies approximately the equation (4); deviation appear only for small n. Since

$$\tilde{\Phi}(x, x_0; n) = \frac{1}{\lambda}\int \frac{\mathrm{d}\lambda}{\lambda}\left(\frac{1}{2\pi i}\right)^l \oint_{c_1} \cdots \oint_{c_l} \prod_{k=1}^{l} \frac{\mathrm{d}\zeta_k}{\zeta_k - \lambda \sum_{i=1}^{l} z(i|k)\zeta_i} \frac{\lambda\zeta_{x_0}}{\zeta_x - \lambda \sum_{i=1}^{l} z(i|x)\zeta_i} ,$$

we have

$$\lambda \sum_{y=1}^{l} z(x|y)\,\tilde{\Phi}(y, x_0) = \int \frac{\mathrm{d}\lambda}{\lambda}\left(\frac{1}{2\pi i}\right)^l \oint_{c_1} \cdots \oint_{c_l} \prod_{k=1}^{l} \frac{\mathrm{d}\zeta_k \zeta_{x_0}}{\zeta_k - \lambda \sum_{i=1}^{l} z(i|k)\zeta_i} \cdot \tag{23}$$

$$\cdot \sum_{y=1}^{l} \frac{\lambda z(x|y)}{\zeta_y - \lambda \sum_i z(i|y)\zeta_i} = \int \frac{\mathrm{d}\lambda}{\lambda}\left(\frac{1}{2\pi i}\right)^l \oint_{c_1} \cdots \oint_{c_l} \sum_{y=1}^{l} \prod_{k\neq y} \frac{\mathrm{d}\zeta_k \zeta_{x_0} \mathrm{d}\zeta_y}{\zeta_k - \lambda \sum_i z(i|k)\zeta_i} \cdot$$

$$\cdot \left[\frac{\partial}{\partial \zeta_x}\left(\frac{1}{\zeta_y - \lambda \sum_i z(i|y)\zeta_i}\right) + \frac{\delta_{xy}}{(\zeta_y - \lambda \sum_i z(i|y)\zeta_i)^2}\right] .$$

The second term in the bracket of (23) may be written

$$\int \frac{\mathrm{d}\lambda}{\lambda}\left(\frac{1}{2\pi i}\right)^l \oint_{c_1} \cdots \oint_{c_l} \prod_{k=1}^{l} \frac{\mathrm{d}\zeta_k \zeta_{x_0}}{\zeta_k - \lambda \sum_i z(i|k)\zeta_i} \frac{1}{\zeta_x - \lambda \sum_i z(i|x)\zeta_i} \cong \tag{24}$$

$$\cong \frac{1}{\lambda}\int \frac{\mathrm{d}\lambda}{\lambda}\left(\frac{1}{2\pi i}\right)^l \oint_{c_1} \cdots \oint_{c_l} \prod_{k=1}^{l} \frac{\mathrm{d}\zeta_k}{\zeta_k - \lambda \sum_i z(i|k)\zeta_i} \frac{\lambda\zeta_{x_0}}{\zeta_x - \lambda \sum_i z(i|x)\zeta_i} = \tilde{\Phi}(x, x_0; n) ,$$

because only the terms belonging to large n are of importance and thus we may replace

$$\int \frac{d\lambda}{\lambda} \sum_n a_n \lambda^n = \sum_n a_n \frac{\lambda^n}{n}$$

by

$$\frac{1}{\lambda} \int d\lambda \sum_n a_n \lambda^n = \sum_n a_n \frac{\lambda^n}{n+1}.$$

The first term in the bracket of (23) vanishes for $x \neq x_0$, because

$$(25) \qquad \oint_{C_1} \dots \oint_{C_l} \sum_{y=1}^{l} \prod_{y \neq k} \frac{d\zeta_k d\zeta_1 \dots \zeta_{x_0}}{\zeta_k - \lambda \sum_i z(i|k)\zeta_i} \frac{\partial}{\partial \zeta_x} \left(\zeta_1 - \lambda \sum_i z(i|y)\zeta_i \right) =$$

$$= \oint_{C_1} \dots \oint_{C_l} \zeta_{x_0} \frac{\partial}{\partial \zeta_x} \left(\prod_{k=1}^{l} \frac{d\zeta_k}{\zeta_k - \lambda \sum_i z(i|k)\zeta_i} \right) = 0$$

as

$$\prod_{k=1}^{l} \frac{1}{\zeta_k - \lambda \sum z(i|k)\zeta_i} = f(\zeta_x)$$

is an one-valued function of ζ_x in the neighbourhood of $\zeta_x = 0$.

Thus, taking into account (24) and (25), we see from (23) that equation (4) is satisfied for $x \neq x_0$.

For $x = x_0$ the first term in bracket of (23) yields

$$-\int \frac{d\lambda}{\lambda} \frac{1}{J}.$$

Thus the right-hand side of (23) may be written, taking into account (7),

$$\frac{1}{\lambda} \int \frac{d\lambda}{\lambda} \frac{\partial}{\partial z(x|x)} \frac{1}{J} - \int \frac{d\lambda}{\lambda} \frac{1}{J} =$$

$$= -\ln \lambda - \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n+1} \left\{ \sum_\mu \prod_j \prod_k z(j|k)^{\mu_{jk}} \frac{\prod_k \mu_k!}{\prod_j \prod_k \mu_{jk}!} \right\} +$$

$$+ \sum_{n=0}^{\infty} \frac{\lambda^n}{n+1} \left\{ \sum_\mu \mu_{xx} \prod_j \prod_k z(j|k)^{\mu_{jk} - \delta_{jx}\delta_{kx}} \frac{\prod_k \mu_k!}{\prod_j \prod_k \mu_{jk}!} \right\} = -\ln \lambda + 1 +$$

$$+ \sum_{n=1}^{\infty} \lambda_n \left[\frac{1}{n+1} \sum_{\Sigma\Sigma \mu_{jk} = n+1} \mu_{xx} \prod_j \prod_k z(j|k)^{\mu_{jk} - \delta_{jx}\delta_{kx}} \frac{\mu_k!}{\prod_j \prod_k \mu_{jk}!} - \frac{1}{n} \sum_{\Sigma\Sigma \mu_{jk} = n} \prod_j \prod_k z(j|k)^{\mu_{jk}} \frac{\mu_k!}{\mu_{jk}!} \right].$$

It is evident that for large n and sufficiently large μ_{xx} the last term may be neglected in comparison to the first. Consequently equation (4) is approximately fulfilled for $x = x_0$ too.

If we substitute in (22b) for J^{-1} the expression (7) and than perform the integration over λ we get an expansion of $\tilde{\Omega}$ in powers of λ; the n-th power coefficient reads

$$\tilde{\omega}(n) = \frac{1}{n+1} \sum_{\mu} \prod_{j=1}^{l} \prod_{k=1}^{l} z(j\,|\,k)^{\mu_{jk}} \frac{\prod_{k} \mu_k!}{\prod_{j} \prod_{k} \mu_{jk}!}, \tag{26}$$

where summation is restricted by (7a). Since

$$\tilde{\psi}(x, x_0;\, n) = \frac{\partial \tilde{\omega}(n)}{\partial z(x_0\,|\,x)},$$

relation (21) follows immediately from (26) by comparing it with (15e). We see also from (12) that our $\tilde{\omega}(n)$ is the first term in the alternating series of $\omega(n)$ divided by the number of terms in this series.

4. – Transition to continuous t-variable.

4·1. – We have shown that the exact solution of an equation of diffusion type is given by (12) and an approximate solution, proper for large n, has the form

$$\tilde{\psi}(x, x_0;\, n) = \frac{1}{n+1} \frac{\partial}{\partial z(x_0\,|\,x)} \left\{ \sum_{\mu} \prod_{j=1}^{l} \prod_{k=1}^{l} z(j\,|\,k)^{\mu_{jk}} \frac{\prod_{k} \mu_k!}{\prod_{j} \prod_{k} \mu_{jk}!} \right\}, \tag{27}$$

where summation is restricted by (7a). This all was done for the case when t-variable takes discrete values. Let us investigate now the case when Δt tends to zero.

We define the function $p(s)$, $s = 0, 1$; $p(0) = b$, $p(1) = a$, where a and b are arbitrary complex numbers. Further we introduce by linear transformation a new variable

$$\xi = \frac{s(a+b) - a}{\sqrt{ab}}.$$

We write $p(s) = q(\xi)$. It may be shown, as usual, with the help of characteristic functions that for

$$P_n(\xi) = \sum_{\xi_1} \dots \sum_{\xi_n} q(\xi_1) \dots q(\xi_n),$$

where the summation is taken over ξ_i satisfying

$$\xi = \frac{1}{\sqrt{n}} (\xi_1 + \ldots + \xi_n) ,$$

a central limit theorem is true; indeed, for large n,

$$P_n(\xi) \cong \frac{(a+b)^{n+1}}{\sqrt{2\pi nab}} \exp\left[- \frac{(a+b)^2}{2nab} \left(\sum_{i=1}^{n} \left(s_i - \frac{a}{a+b} \right) \right)^2 \right], \tag{28}$$

in other words,

$$\frac{n!}{x!\,(n-x)!}\, a^x b^{n-x} \cong \frac{(a+b)^{n+1}}{\sqrt{2\pi nab}} \exp\left[- \frac{(a+b)^2}{2nab} \left(x - \frac{na}{a+b} \right)^2 \right] . \tag{29}$$

We apply this result to perform the computations in (27) for $\Delta t \to 0$. We confine ourselves to the case when the number of levels is bounded or at least the concerned spectrum is discrete, the investigations of the case of continuous spectrum meet some obstacles.

The most suitable case for computations is that of stochastic processes, because all $z(j\,|\,k) \geqslant 0$ and $(a+b) = \sum_{j=1}^{l} z(j\,|\,k) = 1$. If we consider the sum-over-states some complications already arise, although the condition $z(j\,|\,k) \geqslant 0$ holds too. The most troublesome case is, however, that of Quantum Mechanics, because $z(j\,|\,k)$ are complex numbers (*). Let us pay some attention to this case.

On account of (29) we are able to replace for large μ_k

$$\frac{\mu_k!}{\prod_j \mu_{jk}!} \prod_j z(j\,|\,k)^{\mu_{jk}} , \tag{30}$$

by

$$\left(\sum_{j=1}^{l} z(j\,|\,k) \right)^{\mu_k} (2\pi\mu_k)^{-(l-1)/2}\, \sigma^{-1} \cdot \tag{31a}$$

$$\cdot \exp\left[- \frac{1}{2n\sigma^2}\, Q^{-1} \left(\mu_{1k} - \frac{\mu_k\, z(1\,|\,k)}{\sum_j z(j\,|\,k)} , \ldots \mu_{l-1.k} - \frac{\mu_k z(l-1\,|\,k)}{\sum_j z(j\,|\,k)} \right) \right] ,$$

(*) In virtue of (A) and (B) as well as of the continuity of the solution in respect to the t-variable we have for each representation $\sum_{j=1}^{l} z(j\,|\,k) = 1 + 0(\Delta t)$; hence normalization as in case of probabilities holds for $\Delta t \to 0$.

where Q^{-1} is the reciprocal quadratic form to

$$Q(x_1, ..., x_{l-1}) = \sum_{r_1=0}^{2} \sum_{r_2=0}^{2} ... \sum_{r_{l-1}=0}^{2} \sigma_{r_1, r_2, ... r_{l-1}} x_1^{r_1} x_2^{r_2} ... x_{l-1}^{r_{l-1}}$$

and σ^2 is the determinant of the matrix of « dispersion » $\sigma_{r_1, r_2, ... r_{l-1}}$, belonging to the « distribution function » $p(s_1, s_2, ..., s_{l-1})$, $s_1, s_2, ..., s_{l-1} = 0, 1,$

$$p(0, 0, ..., 0) = z(1|k)\left(\sum_{j=1}^{l} z(j|k)\right)^{-1}$$

$$p(1, 0, ..., 0) = z(1|k)\left(\sum_{j=1}^{l} z(j|k)\right)^{-1}$$

$$p(0, 1, ..., 0) = z(2|k)\left(\sum_{j=1}^{l} z(j|k)\right)^{-1}$$

. . . . . .

$$p(0, 0, ..., 1) = z(l-1|k)\left(\sum_{j=1}^{l} z(j|k)\right)^{-1}$$

otherwise $= 0$.

However, (29) enables us to put (30) into another asymptotic form; indeed, we have

$$(31b) \qquad \left(\sum_{j=1}^{l} z(j|k)\right)^{\mu_k} \prod_{j=1}^{l-1} (2\pi n_{j-1,k} a_{jk} b_{jk})^{-\frac{1}{2}} \exp\left[-\sum_{j=1}^{l-1} \frac{(n_{j,k} - n_{j-1,k} b_{jk})^2}{2 n_{j-1,k} a_{jk} b_{jk}}\right],$$

where

$$n_{jk} = \mu_k - \sum_{r=1}^{j} \mu_{rk} \qquad j = 0, 1, ... l-1;\ k = 1, 2, ... l$$

and

$$a_{jk} = \frac{z(j|k)}{\sum_{r=j}^{l} z(r|k)}; \qquad b_{jk} = \frac{\sum_{r=j+1}^{l} z(r|k)}{\sum_{r=j}^{l} z(r|k)},$$

the additional condition (7*a*) reads now

$$\sum_{k=1}^{l} n_{j-1,k} - \sum_{k=1}^{l} n_{j,k} = n_{0j}.$$

The essential feature of both forms (31) is that they have the form of a manifold Gaussian distribution and that the dispersion is proportional to μ_k or

$n_{j-1,k}$ resp. in the exponent. If we go over to continuous t variable we must carry out the transformation

$$\begin{cases} \mu_{jk}\Delta t = t_{jk}\,; \\ \mu_{k}\ \Delta t = t_{k}\ ; \end{cases} \qquad j = 1, \ldots, l-1;\ k = 1, \ldots, l \tag{32}$$

or

$$n_{jk}\Delta t = \tilde{t}_{jk}$$

$$n_{0k}\Delta t = t_{k}$$

resp. Thus we get a new Gaussian distribution in both cases, but with dispersion equal to $t_k\sigma^2\Delta t$ or $t_{j-1,k}a_{j,k}b_{j\,k}\Delta t$ resp. proportional to Δt. It is obvious that, if Δt tends to zero, these distributions will tend to Dirac delta. We shall not, however, replace these expressions straightforward by Dirac delta's; we see that the main contribution in (27) comes from the region

$$t_{jk} \cong \frac{t_k z(j\,|\,k)}{\sum_{r=1}^{l} z(r\,|\,k)} \quad \text{in (31}a\text{)}\,, \tag{33a}$$

$$\tilde{t}_{jk} \cong \tilde{t}_{j-1,k}b_{jk} \quad \text{in (31}b\text{)}. \tag{33b}$$

Since $\sum_j \mu_{jk} = \sum_j \mu_{kj}$ it follows immediately from (33a) and also, after short algebraical calculation, from (33b) that these t_k must fulfil the equations

$$t_j \cong \sum_{k=1}^{l} t_k \frac{z(j\,|\,k)}{\sum_r z(r\,|\,k)}\,, \tag{34}$$

which are already the equations for stationary state,

$$\psi(j; t \to \infty) \cong \frac{t_j}{t}\,;$$

from these equations we can find t_j. Thus, of interest are the contributions due to groups of paths, which show a deviation in regard to the stochastically typical group of paths, leading to stationary state. Computations of these deviations will be the subject of a separate paper; the starting point for these computations will be the equations (12) and (27) as well as (31).

4·2. – We give now some crude estimations for the case of transition probability amplitudes of usual form in x representation, i.e. given by (16) and (15b). What belongs to the factor

$$(\sum_{j=1}^{l} z(j\,|\,k))^{\mu_k}$$

in (31) one gets

$$(\sum_{x'=1}^{l} z(x'\,|\,x))^{\mu_x} \cong \left(\frac{m}{2\pi\hbar\,\Delta t i}\right)^{\mu_x/2}\left\{\int \mathrm{d}x' \exp\left[\frac{mi}{2\hbar\,\Delta t}(x'-x)^2 - \frac{i}{\hbar}V(x')\,\Delta t\right]\right\}^{\mu_x} \cong$$

$$\cong \exp\left[-\frac{i}{\hbar}V(x)\,\Delta t\mu_x\right] = \exp\left[-\frac{i}{\hbar}V(x)t_x\right].$$

Here we have employed the method of stationary phase as well as that

$$\text{(A)} \qquad \int_{-\infty}^{+\infty} \exp\left[\frac{im}{2\hbar\,\Delta t}(x'-x)^2\right]\mathrm{d}x' = \left(\frac{2\pi\hbar\,\Delta t i}{m}\right)^{\frac{1}{2}}.$$

What belongs the solution of (34) for t_x, i.e. the stationary state, so

$$\text{(B)} \qquad z(x'\,|\,x) \cong \delta(x-x')[1 + O(\Delta t)] + (O\Delta t)\,,$$

consequently all t_x are approximately equal, hence e.g.

$$(35) \qquad \prod_{x'=1}^{l} \exp\left[-\frac{i}{\hbar}V(x')\,\Delta t\mu_x\right] \cong \exp\left[-\frac{i}{\hbar}\,t\,\frac{1}{l}\int V(x')\,\mathrm{d}x'\right].$$

These results change from one representation to another; e.g. for k representation we get

$$(\sum_{k'=1}^{l} z(k'\,|\,k))^{\mu_k} \cong \left(\int \mathrm{d}k' \left\langle k' \left| \exp\left[-\frac{i}{\hbar}\,\hat{}\,tH\right| k\right\rangle\right)^{\mu_k} \cong \exp\left[-\frac{i}{\hbar}t_k\left(\frac{\hbar^2k^2}{2m} + V(0)\right)\right].$$

4·3. – In connection with representations we mention that we are able to construct $z(j\,|\,k)$ in any quantum representation, if only the transition probability amplitudes are known in solely one representation

$$\left(\frac{m}{2\pi\hbar\,\Delta t i}\right)^{\frac{1}{2}}\iint \mathrm{d}x'\mathrm{d}x\,\langle a'\,|\,x'\rangle \exp\left[\frac{mi}{2\hbar\,\Delta t}(x'-x)^2 - \frac{i}{\hbar}V(x')\,\Delta t\right]\langle x\,|\,a\rangle = z(a'\,|\,a)\,.$$

4·4. – Further we mention that the idea, presented in this note, may be immediately generalized from one to several dimensions. To this aim the

indices j and k in $z(j|k)$ as well as in m_{jk} must be replaced by vectors, e.g.

$$\boldsymbol{j} = (j_1, ..., j_f), \qquad \boldsymbol{k} = (k_1, ..., k_f).$$

5. – The usual perturbation calculus.

If the transition probability amplitudes have the shape of (16) and (15*b*) and if $V\Delta t\hbar^{-1} \ll 1$ or tends to infinity for considered set of states, as e.g. in the case of helium, it may be shown that the application of the method of slow varying factor in the μ_{jk}-space leads to results which are obtained by means of the usual perturbation method; we must, of course, work in the Schrödinger picture, where interaction energy is independent of time. This new approach to the known solution of perturbation method may call some attention; perhaps other mathematical conditions as usual are required for the approximation to be valid.

Finally we mention that from the crude result (35) one may conclude that the first approximation step of the perturbation method yields the solution in form of a product of the undisturbed solution and the unitary factor given by the right hand side of (35), which is of course physically meaningless. We get rid of this factor by normalizing the potential $V(x)$ in such a way that

$$\int V(x')\,\mathrm{d}x' = 0\,.$$

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Elementary Particle Reactions.

C. O. Beasley, Jr. and W. G. Holladay
Physics Department, Vanderbilt University - Nashville, Tennessee

(ricevuto il 20 Luglio 1957)

Contents. – **1.** Introduction. – **2.** Algebra of elementary particle reactions. – **3.** Threshold calculations. – **4.** Conclusion.

1. – Introduction.

When an elementary particle, such as a proton, meson, etc., strikes another particle or group of particles, one or more new particles may be created in the process. For a given incident particle and a given target particle, particles chosen at random may or may not be produced. For one thing, very small cross sections are connected with the direct production of neutrinos, electrons, and muons as well as with the production of photons in many instances, so we will be solely concerned here with the production of particles more massive than muons. In addition, if the total rest mass of the product particles is greater than the total rest mass of the initiating particles, conservation of energy requires that the incident particle have a certain minimum energy, the threshold energy, for the reaction to proceed. It is further presumed that electric charge must be conserved in the reaction. Furthermore, two other quantum numbers [1], called here the « attribute » A and the « heaviness » B, which refer to intrinsic properties of the particles and which add algebraically for a state of more than one particle apparently have to be conserved [2] if the production reaction is to occur with any appreciable cross section. Table I gives the A and B numbers for the known and suspected particles to be considered here.

B, the heaviness quantum number, seems to be rigorously conserved in

[1] M. Gell-Mann: *Phys. Rev.*, **92**, 833 (1953); T. Nakano and K. Nishijima: *Progr. Theor. Phys.*, **10**, 581 (1953); R. G. Sachs: *Phys. Rev.*, **99**, 1573 (1955).

[2] See *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics* (New York, 1956), for a discussion of the experiments relating to conservation of these numbers.

TABLE I. – *The « attribute » A and the « heaviness » B for the class of strongly interacting particles. The bar over a particle indicates an anti-particle. The numbers in the squares are the masses in* GeV *of the particles used in the threshold calculations.*

		A		
		1	0	−1
B	1	PN .939	Λ^0 1.115 $\Sigma^+\Sigma^0\Sigma^-$ 1.189	$\Xi^-\Xi^0$ 1.321
	0	K^+K^0 .493	$\pi^+\pi^0\pi^-$ .139	$K\ \bar{K}^0$ .493
	−1	$\bar{\Xi}\ \bar{\Xi}^0$ 1.321	$\bar{\Lambda}^0$ 1.115 $\bar{\Sigma}^-\bar{\Sigma}^0\bar{\Sigma}^+$ 1.189	$\bar{N}\bar{P}$ .939

all reactions. A, which appears in the expression [3] $q = A/2 + T_3$, where q is the charge of the particle and T_3 its third component of isotopic spin, seems to be conserved in all « fast » reactions such as the collision processes contemplated here, but is not conserved in the « slow » processes among the particles in Table I.

For a given particle being bombarded by a given incident particle, many possible reactions (called here valid reactions) exist consistent with the conservation of the above-mentioned quantum numbers, and, of course, there are many others (called here invalid reactions) which do not conserve these numbers.

From what has been said, it is easy to answer the question as to whether a given set of final particles will be produced from a given set of initial particles. If the total charge, attribute, and heaviness are not conserved, the cross section for the reaction will presumably be so small that the reaction will not be observed, but on the other hand if these quantities are conserved, it is expected that the reaction will occur, provided, of course, that energy-momomentum conservation holds [4].

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[3] A number of people have emphasized the efficacy of this relation. R. G. SACHS: *Phys. Rev.*, **99**, 1573 (1955); B. D'ESPAGNAT and I. PRENTKI: *Nuclear Phys.*, **1**, 33 (1956); J. TIOMNO: *Phys. Rev.*, **103**, 1589 (1956).

[4] Since our initial and final states are in the continuum, no account need be taken of parity and angular momentum conservation. Furthermore, for the particles listed in Table I, conservation of B automatically takes care of the selection rules based on conservation of angular momentum which would forbid, say, an integral and a half-integral spin particle from making two half-integral spin particles.

Not as easy to answer is this question: for a given pair of initial particles, what are all the final sets of particles which may be produced? Even if attention is restricted to those final states involving no more than three or four particles, the possibilities are so numerous that a haphazard attempt to write down all the final state products could very well leave some unsuspected but possible reaction unlisted. In the next section of this paper we provide a systematic procedure for writing down all the possible valid reactions for a given initial pair of particles. The procedure automatically eliminates the invalid ones.

This procedure has been applied to determine the possible reaction products of a nucleon, a pion, photon, electron, kaon, anti-kaon and anti-nucleon incident upon a nucleon. These reaction products are listed in Table III for reactions including as many as four product particles In addition, the threshold energy has been calculated for each of these processes under three different conditions: 1) when the target is a free nucleon; 2) when the target nucleon is in the nucleus, in which the nucleon is considered to have kinetic energy of 25 MeV with momentum toward the incident particle ([5]); 3) by means of a double stage reaction first considered dy FELDMAN ([6]), in the first stage of which a nucleon strikes another nucleon (at rest or in motion in a nucleus) and creates a single high energy pion, which then becomes the incident particle for the second stage of the process. Since in process 3) there are two target nucleons involved, the threshold energy for a certain reaction by way of the double-stage process is lower than that of the single stage process in which an incident nucleon strikes a target nucleon and creates the products directly. This lowering of the threshold is somewhat analogous to that which occurs when the mass of the target is increased; for such an increase there is a decrease in energy lost in center of mass motion.

2. – Algebra of elementary particle reactions.

In this section we wish to provide a general procedure for writing down all valid reactions involving known or suspected particles heavier than muons. A valid reaction is defined to be one which conserves the numbers A and B. We will not distinguish the charge states of a given particle, but if that is done, the procedure can still be carried out.

Each particle is given its designation by its A and B numbers; this designation for a particle is called a term ([7]). For example, the term denoting

([5]) W. MCMILLAN and E. TELLER: *Phys. Rev.*, **72**, 1 (1947).

([6]) G. FELDMAN: *Phys. Rev.*, **95**, 1697 (1953).

([7]) Each term represents a particle which may have different charge states. It should also be pointed out that the (01) and (0—1) terms have further non unique properties since they represent both Λ^0, Σ and Λ^0, $\overline{\Sigma}$ respectively. This fact should be kept explicitly in mind when the techniques presented here are applied.

the nucleon is « 11 », that of the pion « 00 », or that of the K^- particle « 10 ». The A quantum number is the first number of the term, and the B quantum number is the second. There are nine possible terms. We define a set to be any group of these terms. Now each of the nine terms may be expanded into a set of two terms, the sum of whose A and B quantum numbers gives that term which was expanded. This expansion just gives pairs of particles into which a given particle may transform and still conserve the A and B quantum numbers. In Table II we see a listing of all possible expansions of the nine terms into sets of two terms.

TABLE II. – *All possible expansions into sets of two terms.*

$11 \to 11 + 00$	$10 \to 11 + 0-1$	$1-1 \to 10 + 0-1$
$10 + 01$	$10 + 00$	$00 + 1-1$
	$1-1 + 01$	
$01 \to 00 + 01$	$00 \to 00 + 00$	$0-1 \to 0-1 + 00$
$11 + -10$	$0-1 + 01$	$-10 + 1-1$
$10 + -11$	$-10 + 10$	$-1-1 + 10$
	$-1-1 + 11$	
	$-11 + 1-1$	
$11 \to 01 + -10$	$-10 \to -11 + 0-1$	$-1-1 \to 10 + 0-1$
$00 + -11$	$-10 + 00$	$00 + -1-1$
	$-1-1 + 01$	

Now suppose we start with a given initial pair of particles. First, all valid sets of two product particles are easily written down by methods of exhaustion. Now we wish to find all valid reactions in which the final state consists of three particles. To do this, we take a term of one of our sets of two product terms and expand it by using Table II. This expansion of a single term, plus our unexpanded term, gives us a set of three terms. To assure that all valid reactions are obtained and that duplications will not arise, the following rules have been formed for the expansion of a group of n-term sets into a gronp of $n+1$-term sets. These rules are stated here without proof, but a proof of them may be found in the Appendix. These are the rules:

1) All valid reactions in which $n+1$ particles are produced from a given pair of initial particles can be formed by expanding all the different terms of the valid sets of n terms.

2) In the expansion of the terms of a set, the expansion of the first term being expanded in that set may contain a 00 term in its expansion; thereafter, to avoid duplication, no expansion of any of the remaining terms of that set may contain a 00 term.

3) A set of $n+1$ terms from the expansion of a set of n terms has already appeared if $n-1$ terms of the set of $n+1$ terms are present in one of the sets of n terms which has been previously expanded.

We shall use as an illustration of the procedure a nucleon-nucleon (N-N) reaction. In this case, there is no valid reaction which leads to a final state of two particles different from the initial state. To obtain the three-term sets, we note that there is only one term to expand, since both of our only set of two terms are 11's. Therefore, all valid reactions with three product particles are

$$11 + 11 + 00 \qquad 11 + 10 + 01 \tag{1}$$

These represent the product particles of the reactions

$$\mathrm{N} + \mathrm{N} \to 2\mathrm{N} + \pi \qquad \text{and} \qquad \mathrm{N} + \mathrm{N} \to \mathrm{N} + \mathrm{K} + \Lambda^0 \to \mathrm{N} + \mathrm{K} + \Sigma\,. \tag{2}$$

Now we have all valid reactions with three product particles. From the sets listed in (1), we may obtain all valid reactions for four product particles by holding two of the three terms fixed and expanding the third according to the same expansion rules. In our example, we see from our two sets of three terms that there are five different terms to be expanded in order to get all possible sets of four terms. In the expansion of the 11 term, in the first set of (1), we get

$$11 + 11 + 00 + 00 \qquad 11 + 00 + 10 + 01\,, \tag{3}$$

and in the expansion of the 00 term, we get

$$\begin{cases} 11 + 11 + 0 - 1 + 01 & 11 + 11 + -1 - 1 + 11 \\ 11 + 11 + -10 + 10 & 11 + 11 + -11 + 1 - 1\,. \end{cases} \tag{4}$$

Notice that one term of the expansion of 00 was left out because of the application of rule number two. Now in the expansion of the 11 term of the second set in (1), we get only one term

$$10 + 01 + 10 + 01 \tag{5}$$

the other having dropped out under rule number three. Likewise, the expansions of the 10 and 01 terms have only one term each; all other terms are locked out by rule number three. The two unique terms are

$$11 + 01 + 1 - 1 + 01\,, \qquad 11 + 10 + -11 + 10\,. \tag{6}$$

Thus the nine sets in (3), (4), (5) and (6) are all of the valid reactions with four product particles. The procedure may be extended to include the 5-particle product reactions and so on.

3. – Threshold calculations.

In this section we record the formulas used in calculating the threshold kinetic energy T_2 in the laboratory system for the various processes. If E_1 is the total laboratory energy of the target nucleon with mass M_1 which is assumed to be moving directly toward the projectile which has mass M_2, and if M_i is the mass of the i-th one of n product particles, then straightforward relativistic mechanics with conservation of energy and momentum yields

$$T_2 = \frac{AE_1 - [(A^2 - M_1^2M_2^2)(E_1^2 - M_1^2)]^{\frac{1}{2}}}{M_1^2} - M_2 , \tag{7}$$

where $A = \frac{1}{2}[(\sum_{i=1}^{n} M_i)^2 - M_1^2 - M_2^2]$. Here, $c=1$. If the target nucleon is at rest, then $E_1 = M_1$, and

$$T_2 = \frac{1}{2M_1}[(\sum_i M_i)^2 - (M_1 + M_2)^2] . \tag{8}$$

From Eq. (7) and (8) all threshold energies for a single stage process can be calculeted. However, as mentioned in the Introduction, there is a double stage process which yields somewhat lower thresholds than those for the single stage nucleon-nucleon process. In this reaction, a nucleon strikes another nucleon creating a pion which in turn creates the new particles. The threshold energy of the pion-induced reaction can be calculeted by Eq. (7) and (8). Then, the problem remains to find what minimum energy a nucleon must have to give a pion of this energy.

If (E_1, P_1), (E_2, P_2) and (E_π, P_π) are the total energy and momentum for the target nucleon, the projectile nucleon, and the produced pion, this minimum energy is given by

$$E_2 = \frac{Q}{2}(1 + 3S) + \frac{R}{2}[(1 + 9S)(1 + S)]^{\frac{1}{2}} , \tag{9}$$

where $Q = E_\pi - E_1$, $R = P_\pi + P_1$, $S = M^2/R^2 - Q^2$, and M = nucleon mass.

From Eqs. (7), (8) and (9) thresholds for all sets of two, three, and four product particles have been calculated for the types of projectiles and targets mentioned in the Introduction. These thresholds are listed in Table III in order of increasing energy within each set.

TABLE III. – *Reaction products and threshold kinetic energies* (GeV) *for various initial pairs of particles. The column labeled* N *refers to a target nucleon at rest. The column labeled* $\bar{N}$ *refers to target nucleon with* 25 MeV *of energy moving toward projectile.*

Reaction products	$\gamma+N$ $e+N$ (*)		$\pi+N$		$N+N \to 2N+\pi$ $\pi+N$		$N+\bar{N} \to 2N+\pi$ $\pi+N$	
	N	$\bar{N}$	N	$\bar{N}$	N	$\bar{N}$	N	$\bar{N}$
$N+\pi$	.15	.12			.29	.29	.14	.14
$K+\Lambda^0$	.91	.72	.76	.58	1.13	0.94	0.96	0.76
$K+\Sigma$	1.04	.82	.89	.68	1.27	1.02	1.08	0.87
$N+2\pi$	.32	.26	.17	.11	.46	.39	0.32	0.25
$K+\Lambda^0+\pi$	1.15	.91	1.01	.77	1.39	1.13	1.20	0.96
$K+\Sigma+\pi$	1.29	1.03	1.15	.88	1.53	1.25	1.35	1.07
$N+\bar{K}+K$	1.50	1.19	1.36	1.05	1.76	1.45	1.56	1.26
$2K+\Xi$	2.36	1.87	2.20	1.73	2.63	2.15	2.43	1.97
$2N+\bar{N}$	3.75	2.98	3.61	2.83	4.10	3.30	3.9	3.0
$N+\Lambda^0+\bar{\Lambda}^0$	4.88	3.87	4.73	3.72	5.3	4.3	5.1	4.0
$N+\Lambda^0+\Sigma$	5.13	4.07	4.98	3.92	5.5	4.5	5.3	4.2
$N+\Sigma+\bar{\Lambda}^0$	5.13	4.07	4.98	4.07	5.5	4.5	5.3	4.2
$N+\Sigma+\Sigma$	5.38	4.27	5.24	4.12	5.7	4.6	5.6	4.4
$2\Lambda^0+\bar{\Xi}$	6.24	4.95	6.10	4.81	6.6	5.3	6.4	5.1
$N+\Xi+\Xi$	6.35	5.04	6.21	4.89	6.7	5.4	6.5	5.2
$\Lambda^0+\Sigma+\bar{\Xi}$	6.53	5.17	6.38	5.02	6.9	5.5	6.7	5.3
$2\Sigma+\bar{\Xi}$	6.82	5.40	6.67	5.25	7.2	5.7	7.0	5.6
$N+3\pi$	.51	.40	.36	.26	0.68	0.57	.52	.41
$K+\Lambda^0+2\pi$	1.41	1.12	1.27	.98	1.66	1.36	1.48	1.18
$K+\Sigma+2\pi$	1.57	1.24	1.42	1.10	1.81	1.48	1.62	1.30
$K+N+\pi+K$	1.79	1.42	1.65	1.28	2.07	1.68	1.86	1.49
$2K+\pi+\Xi$	2.71	2.15	2.56	2.01	3.0	2.4	2.8	2.2
$2K+\Lambda^0+\bar{K}$	3.11	2.47	2.97	2.33	3.4	2.8	3.2	2.6
$2K+\Sigma+\bar{K}$	3.32	2.63	3.17	2.49	3.6	2.9	3.4	2.7
$2N+\pi+\bar{N}$	4.18	3.31	4.03	3.16	4.5	3.6	4.3	3.4
$N+\pi+\Lambda^0+\bar{\Lambda}^0$	5.36	4.25	5.20	4.09	5.7	4.6	5.5	4.3
$N+\pi+\Sigma+\bar{\Lambda}^0$	5.61	4.45	5.46	4.31	6.0	4.8	5.8	4.6
$N+\pi+\Lambda^0+\bar{\Sigma}$	5.61	4.45	5.46	4.31	6.0	4.8	5.8	4.6
$N+\Sigma+\bar{\Sigma}+\pi$	5.88	4.66	5.73	4.53	6.2	5.0	6.0	4.8
$N+\Lambda^0+\bar{N}+K$	5.99	4.76	5.85	4.62	6.4	5.1	6.2	4.9
$2N+\bar{\Lambda}^0+\bar{K}$	5.99	4.76	5.85	4.62	6.4	5.1	6.2	4.9
$N+\Sigma+\bar{N}+K$	6.27	4.98	6.13	4.83	6.6	5.3	6.4	5.1
$2N+\bar{\Sigma}+\bar{K}$	6.27	4.98	6.13	4.83	6.6	5.3	6.4	5.1
$2\Lambda^0+\bar{\Xi}+\pi$	6.77	5.37	6.62	5.23	7.1	5.7	6.9	5.5
$N+\Xi+\bar{\Xi}+\pi$	6.88	5.46	6.74	5.31	7.2	5.8	7.1	5.6
$\Lambda^0+\Sigma+\bar{\Xi}+\pi$	7.07	5.60	6.92	5.46	7.4	6.0	7.2	5.8
$2\Sigma+\bar{\Xi}+\pi$	7.36	5.84	7.21	5.69	7.7	6.2	7.5	6.0
$2\Lambda^0+\bar{\Lambda}^0+K$	7.36	5.84	7.22	5.69	7.7	6.2	7.5	6.0
$N+\Xi+\bar{\Lambda}^0+K$	7.49	5.94	7.35	5.80	7.9	6.3	7.7	6.1

(*) An electron should be included as a reaction product for this reaction.

TABLE III (*continued*).

Reaction products	$\gamma+N$ $e+N$ (*)		$\pi+N$		$N+N \to 2N+\pi$ $\pi+N$		$N+\bar{N} \to 2N+\pi$ $\pi+N$	
	N	$\bar{N}$	N	$\bar{N}$	N	$\bar{N}$	N	$\bar{N}$
$N+\Lambda^0+\bar{\Xi}+K$	7.49	5.94	7.35	5.80	7.9	6.3	7.7	6.1
$\Lambda^0+\Sigma+\bar{\Lambda}^0+K$	7.68	6.09	7.53	5.93	8.0	6.4	7.9	6.2
$2\Lambda^0+\Sigma+K$	7.68	6.09	7.53	5.93	8.0	6.4	7.9	6.2
$N+\Xi+\Sigma+K$	7.8	6.2	7.66	6.05	8.1	6.5	8.0	6.4
$N+\Sigma+\bar{\Xi}+\bar{K}$	7.8	6.2	7.66	6.05	8.1	6.5	8.0	6.4
$2\Sigma+\Lambda^0+K$	8.0	6.3	7.8	6.2	8.3	6.7	8.1	6.5
$\Lambda^0+\Sigma+\Sigma+K$	8.0	6.3	7.8	6.2	8.3	6.7	8.1	6.5
$2\Sigma+\Sigma+K$	8.3	6.6	8.2	6.4	8.7	6.9	8.5	6.7
$\Lambda^0+\Xi+\bar{\Xi}+K$	9.1	7.3	9.0	7.1	9.5	7.6	9.3	7.4
$\Sigma+\Xi+\Xi+K$	9.5	7.5	9.3	7.4	9.8	7.9	9.6	7.7

Reaction products	$N+\bar{N}$		Reaction products	$K+N$	
	N	$\bar{N}$		N	$\bar{N}$
$2N+\pi$	0.29	0.14	$N+K+\pi$	0.22	0.12
$N+\Lambda^0+K$	1.58	1.10	Λ^0+2K	1.26	0.91
$N+\Sigma+K$	1.78	1.26	$\Sigma+2K$	1.43	1.05
$2N+2\pi$	0.59	0.35	$2N+\bar{\Lambda}^0$	3.68	2.82
$N+\Lambda^0+K+\pi$	1.96	1.40	$2N+\bar{\Sigma}$	3.91	3.01
$N+\Sigma+K+\pi$	2.17	1.57	$N+\Lambda^0+\bar{\Xi}$	4.97	3.85
$2N+K+K$	2.49	1.81	$N+\Sigma+\bar{\Xi}$	5.24	4.06
$2\Lambda^0+2K$	3.63	2.70	$N+K+2\pi$	0.46	0.29
			$\Lambda^0+2K+\pi$	1.58	1.17
			$\Sigma+2K+\pi$	1.76	1.31
$N+2K+\Xi$	3.73	2.78	$N+2K+K$	2.03	1.52
$2K+\Lambda^0+\Sigma$	3.88	2.91	$\Xi+3K$	3.09	2.36
$2K+2\Sigma$	4.15	3.11	$2N+\bar{\Lambda}^0+\pi$	4.12	3.18
$3N+\bar{N}$	5.63	4.29	$2N+\Sigma+\pi$	4.37	3.37
$2N+\Lambda^0+\bar{\Lambda}^0$	7.10	5.45	$2N+N+K$	4.75	3.67
$2N+\Lambda^0+\Sigma$	7.43	5.72	$N+\Lambda^0+\Xi+\pi$	5.47	4.25
$2N+\Sigma+\bar{\Lambda}^0$	7.43	5.72	$N+\Xi+\Sigma+\pi$	5.76	4.47
$2N+\Sigma+\Sigma$	7.76	5.98	$N+\Lambda^0+\bar{\Lambda}^0+K$	6.05	4.71
$N+2\Lambda^0+\Xi$	8.9	6.8	$2N+\bar{\Xi}+\bar{K}$	6.17	4.79
$2N+\Xi+\Xi$	9.0	7.0	$N+\Lambda^0+\bar{\Sigma}+K$	6.35	4.94
$N+\Lambda^0+\Sigma+\Xi$	9.2	7.1	$N+\Sigma+\bar{\Lambda}^0+K$	6.35	4.94
$N+2\Sigma+\Xi$	9.6	7.4	$N+\Sigma+\bar{\Sigma}+K$	6.65	5.17
			$2\Lambda^0+\bar{\Xi}+K$	7.61	5.95
			$N+\Xi+\Xi+K$	7.75	6.05
			$\Lambda^0+\Sigma+\Xi+K$	7.94	6.21
			$2\Sigma+\Xi+K$	8.27	6.46

(*) An electron should be included as a reaction product for this reaction.

TABLE III (*continued*).

Reaction products	K+N	
	N	$\bar{N}$
$\Lambda^0+\pi$	Exothermic	
$\Sigma+\pi$	Exothermic	
$\Xi+K$	.66	.45
$\Lambda^0+2\pi$	Exothermic	
$\Sigma+2\pi$	Exothermic	
$K+N+\pi$	.22	.12
$\Xi+K+\pi$	.93	.66
Λ^0+K+K	1.26	.92
$\Sigma+K+K$	1.43	1.05
$N+\Lambda^0+\bar{N}$	3.68	2.83
$\Sigma+N+N$	3.92	3.02
$2\Lambda^0+\bar{\Lambda}^0$	4.87	3.77
$N+\Xi+\bar{\Lambda}^0$	4.97	3.86
$2\Lambda^0+\Sigma$	5.13	3.98
$\Lambda^0+\bar{\Lambda}^0+\Sigma$	5.13	3.98
$N+\Xi+\Sigma$	5.24	4.07
$\Lambda^0+2\Sigma$	5.40	4.21
$\Lambda^0+\Sigma+\bar{\Sigma}$	5.40	4.21
$2\Sigma+\Sigma$	5.68	4.42
$\Lambda^0+\Xi+\Xi$	6.42	5.01
$\Sigma+\Xi+\bar{\Xi}$	6.72	5.25
$\Lambda^0+3\pi$	.15	.07
$\Sigma+3\pi$	.27	.15
$K+N+2\pi$	.46	.29
$K+\Xi+2\pi$	1.23	.80
$K+K+\Lambda^0+\pi$	1.57	1.16
$K+K+\Sigma+\pi$	1.75	1.31
$2K+K+N$	2.02	1.52
$2K+K+\Xi$	3.08	2.36

Reaction products	K+N	
	N	$\bar{N}$
$N+\Lambda^0+N+\pi$	4.12	3.18
$N+\Sigma+N+\pi$	4.37	3.38
$2N+\bar{N}+K$	4.74	3.65
$2\Lambda^0+\bar{\Lambda}^0+\pi$	5.36	4.17
$N+\Xi+\bar{\Lambda}^0+\pi$	5.47	4.26
$2\Lambda^0+\Sigma+\pi$	5.64	4.39
$\Lambda^0+\Lambda^0+\Sigma+\pi$	5.64	4.39
$N+\Xi+\bar{\Sigma}+\pi$	5.75	4.48
$2\Sigma+\Lambda^0+\pi$	5.92	4.60
$\Lambda^0+\Sigma+\bar{\Sigma}+\pi$	5.92	4.60
$N+\Lambda^0+\bar{\Lambda}^0+K$	6.05	4.71
$2\Lambda^0+N+K$	6.05	4.71
$N+\Xi+N+K$	6.17	4.80
$2\Sigma+\Sigma+\pi$	6.21	4.84
$K+N+\Lambda^0+\bar{\Sigma}$	6.34	4.94
$K+N+\bar{\Lambda}^0+\Sigma$	6.34	4.94
$K+N+\Lambda^0+\Sigma$	6.34	4.94
$K+N+\Sigma+\Sigma$	6.64	5.18
$K+N+2\Sigma$	6.64	5.18
$\Lambda^0+\Xi+\Xi+\pi$	6.98	5.45
$\Sigma+\Xi+\Xi+\pi$	7.29	5.70
$K+2\Lambda^0+\Xi$	7.61	5.96
$K+\Lambda^0+\bar{\Lambda}^0+\Xi$	7.61	5.96
$K+N+\Xi+\Xi$	7.74	6.06
$K+\Lambda^0+\Sigma+\bar{\Xi}$	7.94	6.21
$K+\Lambda^0+\Sigma+\Xi$	7.94	6.21
$K+\Lambda^0+\Sigma+\Xi$	7.94	6.21
$K+2\Sigma+\Xi$	8.26	6.47
$K+\Sigma+\Sigma+\Xi$	8.26	6.47
$K+2\Xi+\Xi$	9.48	7.43

Reaction products	$\bar{N}$+N	
	N	$\bar{N}$
$\Lambda^0+\bar{\Lambda}^0$	0.77	0.48
$\Sigma+\Lambda^0$	0.95	0.62
$\Lambda^0+\Sigma$	0.95	0.62
$\Sigma+\Sigma$	1.13	0.76
$\Xi+\Xi$	1.84	1.30
$N+N+\pi$	0.29	0.14

Reaction products	$\bar{N}$+N	
	N	$\bar{N}$
$\Lambda^0+\bar{\Lambda}^0+K+K$	3.63	2.70
$N+\Xi+2K$	3.73	2.78
$\Xi+N+2K$	3.73	2.78
$\Sigma+\Lambda^0+K+K$	3.88	2.91
$\Lambda^0+\Sigma+K+K$	3.88	2.91
$\Sigma+\Sigma+K+K$	4.15	3.11

TABLE III (*continued*).

Reaction products	$\bar{N}+N$		Reaction products	$\bar{N}+N$	
	N	$\bar{N}$		N	$\bar{N}$
$\Lambda^0+\bar{\Lambda}^0+\pi$	1.11	0.74	$\Xi+\bar{\Xi}+K+K$	5.13	3.90
$\Sigma+\bar{\Lambda}^0+\pi$	1.30	0.88	$2N+2\bar{N}$	5.63	4.38
$\Lambda^0+\bar{\Sigma}+\pi$	1.30	0.88	$N+\Lambda^0+\bar{N}+\bar{\Lambda}^0$	7.10	5.45
$\Sigma+\bar{\Sigma}+\pi$	1.49	1.03	$N+\Lambda^0+\bar{N}+\Sigma$	7.43	5.72
$\Lambda^0+\bar{N}+K$	1.58	1.10	$N+\Sigma+\bar{N}+\bar{\Lambda}^0$	7.43	5.72
$N+\bar{\Lambda}^0+\bar{K}$	1.58	1.10	$N+\Sigma+\bar{N}+\Sigma$	7.76	5.98
$\Sigma+\bar{N}+K$	1.78	1.24	$2\Lambda^0+2\bar{\Lambda}^0$	8.7	6.7
$N+\bar{\Sigma}+\bar{K}$	1.78	1.24	$2\Lambda^0+\bar{N}+\bar{\Xi}$	8.9	6.8
$\Xi+\bar{\Xi}+\pi$	2.23	1.61	$N+\Xi+2\bar{\Lambda}^0$	8.9	6.8
$\Xi+\bar{\Lambda}^0+K$	2.69	1.97	$N+\Xi+\bar{N}+\bar{\Xi}$	9.0	7.0
$\Lambda^0+\bar{\Xi}+K$	2.69	1.97	$2\Lambda^0+\bar{\Lambda}^0+\bar{\Sigma}$	9·1	7.0
$\Xi+\bar{\Sigma}+K$	2.93	2.16	$\Lambda^0+\Sigma+2\bar{\Lambda}^0$	9.1	7.0
$\Sigma+\bar{\Xi}+\bar{K}$	2.93	2.16	$N+\Xi+\bar{\Lambda}^0+\bar{\Sigma}$	9.2	7.1
$2K+2\bar{K}$	0.19	.077	$\Lambda^0+\Sigma+N+\Xi$	9.2	7.1
$N+\bar{N}+2\pi$	0.59	0.35	$2\Lambda^0+2\bar{\Sigma}$	9.4	7.3
$\Lambda^0+\bar{\Lambda}^0+2\pi$	1.46	1.01	$\Lambda^0+\Sigma+\bar{\Lambda}^0+\bar{\Sigma}$	9.4	7.3
$\Lambda^0+\bar{\Sigma}+2\pi$	1.66	1.17	$2\Sigma+2\bar{\Lambda}^0$	9.4	7.3
$\Sigma+\bar{\Lambda}^0+2\pi$	1.66	1.17	$N+\Xi+2\Sigma$	9.6	7.4
$\Sigma+\bar{\Sigma}+2\pi$	1.87	1.33	$2\Sigma+N+\Xi$	9.6	7.4
$\Lambda^0+\bar{N}+K+\pi$	1.96	1.40	$\Lambda^0+\Sigma+2\Sigma$	9.8	7.6
$N+\bar{\Lambda}^0+\bar{K}+\pi$	1.96	1.40	$2\Sigma+\bar{\Lambda}^0+\bar{\Sigma}$	9.8	7.6
$\Sigma+\bar{N}+K+\pi$	2.17	1.57	$2\Sigma+2\Sigma$	10.2	7.9
$N+\bar{\Sigma}+\bar{K}+\pi$	2.17	1.57	$\Lambda^0+\Xi+\bar{\Lambda}^0+\bar{\Xi}$	10.8	8.3
$N+\bar{N}+K+\bar{K}$	2.49	1.81	$\Lambda^0+\Xi+\Sigma+\Xi$	11.1	8.7
$\Xi+\bar{\Xi}+2\pi$	2.65	1.94	$\Sigma+\Xi+\bar{\Lambda}^0+\bar{\Xi}$	11.1	8.7
$\Xi+\bar{\Lambda}^0+K+\pi$	3.13	2.32	$\Sigma+\Xi+\Sigma+\bar{\Xi}$	11.5	9.0
$\Lambda^0+\bar{\Xi}+\bar{K}+\pi$	3.13	2.32	$2\Xi+2\bar{\Xi}$	13.0	10.1
$\Xi+\bar{\Sigma}+K+\pi$	3.37	2.50			
$\Sigma+\bar{\Xi}+\bar{K}+\pi$	3.37	2.50			

(*) The exothermic reactions $\bar{N}+N \rightarrow$ up to 13 pions (inclusive) and $\bar{N}+N \rightarrow \bar{K}+K$ up to 6 pions (inclusive) are not listed.

Fig. 1, 2 and 3 give an easy means of finding approximate thresholds if the total mass of the product particles is known. Each curve is labeled by the type of initiating process. Representative reaction products are given on each curve. These graphs would be useful in determining thresholds for reactions involving more than four product particles which are not listed in Table III.

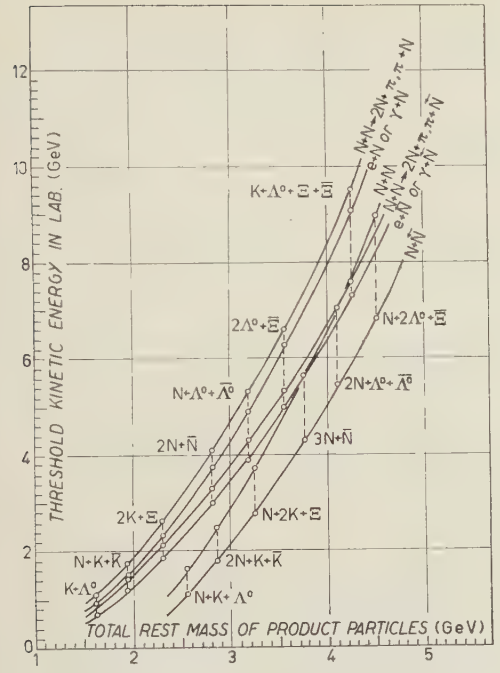

Fig 1.

Figg. 1, 2, 3. – Threshold kinetic energy as a function of the total mass of the reaction products. Each curve is labeled by the projectile and the target. The symbol N represents a target nucleon with 25 MeV of energy with momentum toward the projectile. Representative reaction products are given on each curve. A bar over a particle represents an anti-particle.

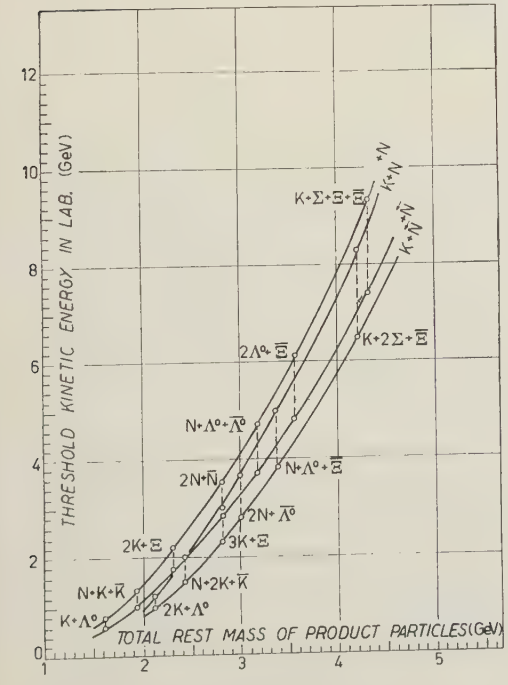

Fig. 2.

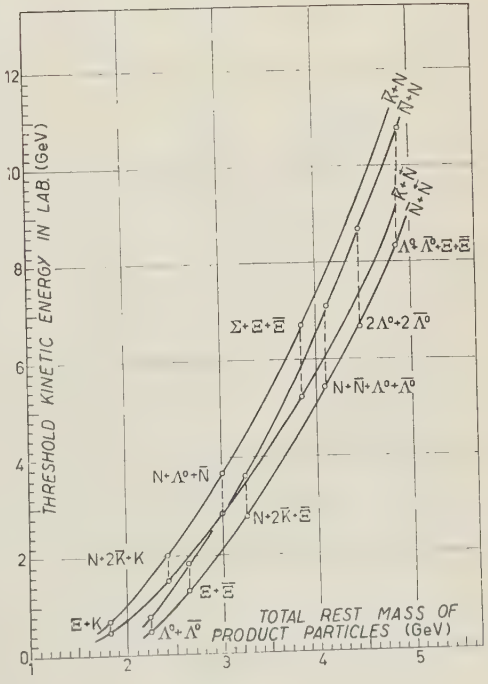

Fig. 3.

Table IV gives a brief listing of the lowest threshold for the production of any elementary particle for a given production process. For each particle, other product particles (with the exception of an arbitrary number of nucleons, a number which depends on the number of nucleons in the initial state of the process) produced with that particle are listed.

TABLE IV. – *Lowest thresholds for a given elementary particle for a given process. Bars over letters indicate anti-particles. An arrow over an* N *means that the target nucleon is in a nucleus and has* KE *of* 25 MeV *with momentum toward projectile.*

Particle	Reaction products (*)	N+N	N+$\overleftarrow{\bar{N}}$	γ+N e+N (+)	γ+$\overleftarrow{\bar{N}}$ e+N (+)	N+N → → 2N+π π+N	N+N → → 2N+π π+$\overleftarrow{\bar{N}}$	N+$\overleftarrow{\bar{N}}$ → → 2N+π π+N	N+$\overleftarrow{\bar{N}}$ → → 2N+π π+$\overleftarrow{\bar{N}}$
K	K+Λ^0	1.58	1.10	0.91	0.72	1.13	0.94	0.96	0.76
$\bar{K}$	K+$\bar{K}$	2.49	1.81	1.50	1.19	1.76	1.45	1.56	1.26
$\bar{N}$	N+$\bar{N}$	5.65	4.29	3.75	2.98	4.1	3.3	3.9	3.0
Λ^0	K+Λ^0	1.58	1.10	0.91	0.72	1.13	0.94	0.96	0.76
$\bar{\Lambda}^0$	Λ^0+$\bar{\Lambda}^0$	7.1	5.45	4.88	3.87	5.3	4.3	5.1	4.0
Σ	K+Σ	1.78	1.26	1.04	0.82	1.27	1.02	1.08	0.87
$\bar{\Sigma}$	Λ^0+$\bar{\Sigma}$	7.4	5.7	5.13	4.07	5.5	4.5	5.3	4.2
Ξ	2K+Ξ	3.73	2.78	2.36	1.87	2.63	2.15	2.43	1.97
Ξ	2Λ^0+$\bar{\Xi}$	8.9	6.8	6.2	4.95	6.6	5.3	6.4	5.1

(*) Nucleons as a reaction product are not listed
(+) An electron is considered to be a reaction product of this reaction.

4. – Conclusion.

On the assumption that the quantum numbers A and B are conserved, a procedure has been devised for writing down all the reaction products composed of particles in Table I. Table III lists these reaction products up to four particles together with the production threshold when the projectile is a photon, electron, pion, kaon, anti-kaon, nucleon or anti-nucleon and the target is a stationary nucleon or a 25 MeV nucleon with momentum toward the projectile. Threshold energies have also been calculated for the Feldman two stage process.

As expected, lowest thresholds for the production of any given elementary particles are associated with the e+N or γ+N process, although the difference of thresholds of that process and the two stage process becomes less marked at high energies. The two stage process lowers the threshold over the analogous one-stage by roughly 20% to 25%. Furthermore, in the single stage process, the threshold for a 25 MeV target nucleon is some 25% lower than that for a target nucleon at rest.

It seems worthwhile to point out that the most massive anti-hyperon, the Ξ, which can be made by the process, energy $+\mathrm{N} \rightarrow 2\Lambda^0 + \Xi$, has a threshold by two the stage process of about 5.3 GeV, and therefore presumably can be made at Berkeley.

Appendix

The rules stated in Sect. 2 for the expansion of a set of n terms into a set of $n+1$ terms will now be proved. Rule 1 essentially states that any valid set of $n+1$ terms (which represent, of course, $n+1$ particles) can be obtained by the expansion of some one term of one of the sets of n terms.

Analytically, what is meant by a valid set of $n+1$ terms is that

$$\sum_{i=1}^{n+1} \alpha_i\beta_i = \sum_{i=1}^{2} \alpha_i'\beta_i' ,$$

where $\alpha_i\beta_i$ is a pair of the integers $(1, 0, -1)$, which, of course, represents one of the particles in Table I, and $\alpha_1'\beta_1'$ and $\alpha_2'\beta_2'$ are the AB designation of the two initial particles. The sum indicates that the α's are to add algebraically as are the β's. Now rule 1 will be true, if some two terms, say $\alpha_n\beta_n + \alpha_{n+1}\beta_{n+1}$, of the first $n+1$ set, combine to give one of the nine possible terms, say ab, representing a particle in Table I, for then

$$\sum_{i=1}^{n+1} \alpha_i\beta_i = \sum_{i=1}^{n-1} \alpha_i\beta_i + ab ,$$

but the expression on the right is a valid set of n terms, such that one of the expansion of the ab term yields our set of $n+1$ terms. Thus, our problem is reduced to one of proving that in a valid set of $n+1$ terms, there are always some two terms which, when added, will give one of the nine terms representing a particle in Table I.

The proof will be subdivided into two cases. In the first case, we suppose that one of the numbers of some term of the $n+1$ set is a zero, say 0β. Now if the second number of that term is also zero, obviously the term will add to any other of the n terms to give a valid particle, since $00 + \alpha_1\beta_1 = \alpha_1\beta_1$. If the second number happens to be a 1, (the proof proceeds in the same way if this number is a -1), then all the remaining n terms cannot have 1's for their second numbers, as the sum of the n terms would give a term with an n in it [8]. Of the initial-state particle set, the one with the greatest sum of A and B quantum numbers, the N-N reaction can only be 22. Hence, one of the remaining n terms must have either a 0 or a -1 for its second number; in either case, that term added to our initial term (01) will result in a valid term.

[8] Throughout we are assuming that all two term sets can be written down by the method of exhaustion and that $n > 2$.

For the second case, we assume that none of the $n+1$ terms has a 0 in it. Now let us refer to Table V below, in which are listed all possible terms without 0's.

TABLE V.

11	1 — 1
— 11	— 1 — 1

Note that the sum of any $n+1$ terms in the same column or same row gives a non valid term with $\pm(n+1)$ in it. Therefore, either the diagonal or anti-diagonal sets of terms, or both, must appear. But it is obvious that the sum of either of the diagonal or anti-diagonal sets of terms gives a valid term, 00. Hence, two of the terms must add in this case to give a valid term.

With regard to the second rule, consider the expansion of the set $\sum_{i=1}^{n} \alpha_i\beta_i$. One expansion of $\alpha_1\beta_1$ is $\alpha_1\beta_1 + 00$; likewise, an expansion of $\alpha_2\beta_2$ is $\alpha_2\beta_2 + 00$, and similarly for the rest of the terms. But any one of these expansions added to the other terms of the set gives $\sum_{i=1}^{n} \alpha_i\beta_i + 00$, so that only one such expansion need be made.

The third rule may be proved by the following line of reasoning. For any given pair of initial particles the sum of terms representing possible reaction products must remain constant. Therefore for a set of n terms

$$(\alpha_1\beta_1, \ldots, \alpha_n\beta_n) , \tag{1}$$

which has been completely expanded into $n+1$ terms, and for another n-term set $(\alpha'_1\beta'_1, \ldots, \alpha'_n\beta'_n)$ which has an expansion

$$(\alpha'_1\beta'_1, \ldots, \alpha'_{n-1}\beta'_{n-1}, a_nb_n + a_{n+1}b_{n+1}) , \tag{2}$$

we have

$$\sum_{i=1}^{n} \alpha_i\beta_i = \sum_{i=1}^{n-1} \alpha'_i\beta'_i + a_nb_n + a_{n+1}b_{n+1} . \tag{3}$$

Now if $\alpha_i\beta_i = \alpha'_i\beta'_i$, when i goes from 1 to $n-1$, then from (3)

$$\alpha_n\beta_n = a_nb_n + a_{n+1}b_{n+1},$$

i.e., $a_nb_n + a_{n+1}b_{n+1}$, is one of the possible expansions of $\alpha_n\beta_n$, and thus the expansion (2) has already appeared as one of the possible expansions of (1). This completes the proof of rule 3.

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Theory of Meson Multiple Production.

D. Ito and H. Tanaka (*)

Tokyo University of Education - Japan
University of Bologna - Bologna, Italy

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Contents. — **1.** Introduction. – **2.** Statistical treatment of the S-matrix. **3.** Perfect gas model (Fermi type theory). – **4.** Imperfect meson gas. – **5.** Hydrodynamical behavior of the meson field. – **6.** Comments on Landau theory. – **7.** A simpler model (Fermi type theory). – **8.** Conclusion. – Appendix A. – Appendix B.

1. – Introduction.

Even before the experimental evidence, the multiple meson production had been discussed in connection with the criticism for applying the limits of the field theoretical description to a high energy region [1], hoping to find there some new conceptions or frames of a future theory [2]. For the moment however, confining ourselves within the frame of the familiar field theory we shall formulate a theory of multiple meson production in a way to clarify the field theoretical meaning of the Fermi [3] and Landau [4] theories proposed rather intuitively.

Recently many authors have treated the multiple meson production in different ways, but their theories may be classified roughly in two types; *a*) theories

(*) On leave of absence from the Yokohama National University, Japan.
(1) W. Heisenberg: *Zeits. f. Phys.*, **101**, 533 (1936); **113**, 61 (1939).
(2) G. Wataghin: *Nuovo Cimento*, **5**, 689 (1957).
(3) E. Fermi: *Progr. Theor. Phys.*, **5**, 570 (1950); *Phys. Rev.*, **81**, 683 (1951).
(4) L. D. Landau: *Izv. Acad. Nauk USSR*, **17**, 51 (1953).

which are formulated in the frame of the field theory, such as those of HEISENBERG, LEWIS-OPPENHEIMER-WOUTHUYSEN [5] (briefly written hereafter as LOW), etc., *b*) theories which seem to be outside of the usual field theory, such as those of FERMI [3-7] and LANDAU [4-8].

Now, before entering into the detailed disscussion of the characters of these theories we shall see first which theories remain valid to explain the recent experimental data in an extremely high energy region such as the multiplicity, the angular distribution or the production branching ratio etc.

a) *Multiplicity.* Heisenberg's non linear theory [9] fails to explain the energy dependence of the multiplicity of produced mesons. For the multiplicity $n_\pi = 20$ measured at the energy 10^{+13} eV the theoretical predictions are given as [10]

$$\langle n_\pi \rangle = 30 \quad \text{(LANDAU)}$$

$$\langle n_\pi \rangle = 11{,}7 \quad \text{(FERMI)}$$

b) *Angular distribution.* The angular distribution of air showers in cosmic rays was not explained by the LOW type theory. The values calculated of the half angle $\theta_{\frac{1}{2}}$, inside of which one half of the energy is emitted, are as following for the energy of 10^{+14} eV,

$$\theta_{\frac{1}{2}} = 6{,}7 \cdot 10^{-4} \quad \text{(FERMI)}$$

$$\theta_{\frac{1}{2}} = 1{,}3 \cdot 10^{-4} \quad \text{(LANDAU)}$$

The experimental value being $1 \cdot 10^{-4}$, FERMI's theory seems to give a rather large value.

c) *Lateral component.* According to NISHIMURA [11] the lateral component $\boldsymbol{k}$ of the meson's momentum in an air shower is almost always of the order of several hundred MeV independently of the incident energy. This fact is rather

[5] H. LEWIS, J. OPPENHEIMER and S. WOUTHUYSEN: *Phys. Rev.*, **73**, 127 (1948).
[6] H. W. LEWIS: *Rev. Mod. Phys.*, **24**, 241 (1952); H. FUKUDA and G. TAKEDA: *Progr. Theor. Phys.*, **5**, 957 (1950); H. UMEZAWA, Y. TAKAHASHI and S. KAMEFUCHI: *Phys. Rev.*, **85**, 505 (1952); T. D. LEE and E. M. HENLEY: *Phys. Rev.*, **101**, 1536 (1956).
[7] J. LEPORE and R. STUART: *Phys. Rev.*, **94**, 1724 (1954); R. H. MILBURN: *Rev. Mod. Phys.*, **27**, 1 (1954); J. KOVACS: *Phys. Rev.*, **101**, 397 (1956).
[8] M. HAMAGUCHI: *Nuovo Cimento*, **4**, 1242 (1956).
[9] W. HEISENBERG: *Zeits. f. Phys.*, **133**, 65 (1952).
[10] L. ROSENTHAL and D. TCHERNAVSKIJ: *Usp. Fiz. Nauk USSR*, **52**, 185 (1954).
[11] J. NISHIMURA: private communication.

difficult to be interpreted by the FERMI theory due to his isotropical equilibrium state.

d) Branching ratio of production. The branching ratio n_{θ^0}/n_{π^0} of θ^0- and π^0-production was measured recently by FUJIMOTO ([12]) ($\sim 0{,}5$). In the FERMI theory this ratio is determined only by the statistical weight of θ^0 and π^0, and results to be $n_{\theta^0}/n_{\pi^0} = 2$. On the other hand KOBA ([13]) showed that the LANDAU theory could yield the value 0,5.

As seen above, the theory of LANDAU seems to represent well, at least in the present stage, the extremely high energy events. So, it will be rather important to analyse the essential characters of the LANDAU theory which gives us an ample scope to see the nature of these phenomena.

First, stimulated by Heisenberg's turbulence theory ([14]) FERMI had developed the theory of multiple production as an evaporation of a perfect gas from the 'hot spot' in thermal equilibrium created in a contracted volume of meson cloud around the colliding nucleons. Then, taking into account the meson-meson interaction in the contracted volume, LANDAU regarded the production process as a jet of ideal fluid described by the 4-dimensional velocities $u_1(rt) \ldots u_4(rt)$, energy density $\varepsilon(rt)$ and pressure $p(rt)$. The behaviour of the meson field is determined by the hydrodynamical equation

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = 0 \,, \tag{1.1}$$

where the energy-momentum tensor is given by

$$T_{\mu\nu} = (\varepsilon + p) u_\mu u_\nu + \delta_{\mu\nu} p$$

and further, the STEFAN-BOLTZMANN relation is supposed a priori, as a state equation

$$p = \tfrac{1}{3} \varepsilon \,. \tag{1.3}$$

As such, a theory of FERMI-LANDAU type appears as a foreign element in the usual field theory, but nevertheless it describes well the high energy phenomena. If so, in the ultra high energy region other unknown laws are dominant differently from the low energy region? Or, the present field theory may be still valid even in the high energy region but its solutions behave

([12]) Y. FUJIMOTO: Private communication.
([13]) Z. KOBA: *Progr. Theor. Phys.*, **15**, 461 (1956).
([14]) W. HEISENBERG: *Zeits. f. Phys.*, **124**, 628 (1948).

approximately as something like a liquid subtance? In order to obtain some information on the limits of the usual field theory it will be important to examine in which way and in which place the FERMI-LANDAU type descriptions can be situated in the frame of the present field theory.

In this paper, the descriptions of FERMI-LANDAU type theories are derived in a field theoretical manner. Such interpretation may justify, more or less from the field theoretical point of view, the descriptions of FERMI and LANDAU, but not necessarily their artificial pictures such as « thermoequilibrium » or « fluid ». In fact, one can obtain the same features by a more natural model in a statistical treatment of the field theory. In Sect. **2** the general structure of the S-matrices for multiple production are discussed from the statistical point of view. Then, as a special case one obtains in Sect. **3** the perfect gas model (FERMI type theory) where the correspondence with the original FERMI theory becomes clear. Sect. **4** is referred briefly to the case of the imperfect gas model. The hydrodynamical behaviour of the meson field is derived in Sect. **5**, in which the starting equations of the LANDAU theory are obtained. In Sect. **6** some comments on the LANDAU theory are given concerning his initial condition which plays an essential role in determining the characters of the meson fluid. Finally in Sect. **7** a simpler model as in Sect. 3, based on a statistical field theory, is shown to give good agreement with the recent experiments.

2. – Statistical treatment of the S-matrix.

The statistical theory of meson multiple production firstly suggested by HEISENBERG and realized by FERMI has received many criticisms and objections because of its theoretical inconsistencies ([15]), but it plays an important role in the research of the general features of the high energy events. Then the concept of « thermoequilibrium » of FERMI's theory being rather a foreign element in the field theory, some trials were done recently to derive the equilibrium state by treating the S-matrix statistically ([16]).

Now, to understand this idea clearly we shall consider an example of black body radiation. Taking a large hole surrounded by a perfectly reflecting wall

([15]) S. TAKAGI: *Progr. Theor. Phys.*, **7**, 123 (1952); H. FUKUDA: *Phys. Rev.*, **89**, 842 (1951); C. AULUCK and D. KOTHARI: *Phys. Rev.*, **90**, 1002 (1953); S. NANDAU: *Progr. Theor. Phys.*, **11**, 605 (1954); W. KRAUSHAR and L. MARKS: *Phys. Rev.*, **93**, 326 (1954); K. ONO and K. YOKOI: *Progr. Theor. Phys.*, **13**, 101 (1955).

([16]) D. ITO: The essential idea of the statistical treatment of the field theory for meson multiple production was published in Japanese in 1955 and 1956. (See the footnote of the paper of Y. TAKAHASHI) ([16]). Publication in English has been retarded by some trivial reasons. Y. TAKAHASHI: *Nuovo Cimento*, **4**, 531 (1956).

we suppose firstly that in this hole region there exist only monochromatic photon waves of frequency ω_0. This state is represented by an occupation number of photons as $n(\omega) = n(\omega = \omega_0) = n_0$. Then, at the time $t = t_0$ if we throw into this hole an atom which interacts weakly with photons, transitions begin to occur between the different states of photons. So, after a long lapse of time the transition probabilities to the state whose occupation number distribution is $n(\omega) = f(\omega)$ should be estimated by the square of the corresponding S-matrix

$$\text{(2.1)} \qquad \text{Prob}\,[n(\omega) = f(\omega)] \sim |\langle \psi[n(\omega) = f(\omega)],\, S\psi_0[n(\omega) = n_0]\rangle|^2 .$$

Experimentally however, we know that after a sufficiently long time the distribution of photon numbers will be following Planck's law,

$$\text{(2.2)} \qquad n(\omega) = \frac{1}{\exp[\hbar\omega/kT] - 1} \qquad (kT = \hbar\omega_0).$$

This means that the probability has a large value only for the state in which $n(\omega)$ is given by Planck's distribution. In other words, the pratically important state is only $\psi[n(\omega)=\text{Planck's distribution}]$ (*). This state should be called as the most probable final state for radiation.

In the field theory we could obtain, at least in principle, the transition probabilities to any state, apart from the practical difficulties of calculation. But in the real cases the created mesons are so many that the observed final state may be very often in the most probable state or in nearing ones. So, taking only the most probable final state the theory will be remarkably simplified. With such approximation we shall discuss the structure of the S-matrix for multiple production.

The cross section of n pion production by nucleon-nucleon collision is proved to be

$$\text{(2.3)} \qquad \sigma_n = \frac{(2\pi)^4}{v_r}\int \frac{1}{n!}\,|\langle \boldsymbol{p}_1^0 \boldsymbol{p}_2^0 | I | \boldsymbol{p}_1 \boldsymbol{p}_2,\, \boldsymbol{k}_1 \ldots \boldsymbol{k}_n\rangle|^2 \cdot \frac{\mathrm{d}\boldsymbol{p}_1}{E_1}\frac{\mathrm{d}\boldsymbol{p}_2}{E_2}\frac{\mathrm{d}\boldsymbol{k}_1}{\varepsilon_1}\frac{\mathrm{d}\boldsymbol{k}_2}{\varepsilon_2}\ldots\frac{\mathrm{d}\boldsymbol{k}_n}{\varepsilon_n}\,\delta^4(\Delta p - k_1 - k_2 - \ldots - k_n)\,,$$

(*) $\Psi[n(\omega) = 1/(\exp[\hbar\omega_0/kT] - 1)]$ is one of the eigenstates of the Hamiltonian H_0 for the hole radiation system, and satisfies the equation

$$n_0\hbar\omega_0\Psi = \bar{H}_0\Psi$$

$$\because\ \bar{H}_0\Psi = \sum_s \hbar\omega_s n_s \Psi = \sum_s \frac{\hbar\omega_s}{\exp[p(\hbar\omega_s/kT)] - 1}\,\Psi = n_0\hbar\omega_0\Psi\,.$$

where

$\boldsymbol{P}_1^0, \boldsymbol{P}_2^0$: momenta of the incident nucleons

V_r: relative velocity of the incident nucleons

$P_1 \equiv (\boldsymbol{P}_1, E_1)$, $P_2 \equiv (\boldsymbol{P}_2, E_2)$: energy momentum of final nucleons

$k_i \equiv (\boldsymbol{k}_i, \varepsilon_i)$:energy momentum of emitted mesons

$\Delta P = P_1^0 + P_2^0 - P_1 - P_2 \quad (\Delta \boldsymbol{P}, \Delta W)$

$(1/\sqrt{n!}) \langle | I | \rangle$: invariant collision matrix of Møller

Then, the most probable final state is obtained as a solution of the variational problem for the transition probability $|\langle | I | \rangle|^2$. For example in the occupation number representation of the transition probability $P[n_1, n_2 \ldots]$, introducing the entropy S by $P[n_1, n_2, \ldots;] = \exp[S[n_1 n_2 \ldots] \cdot \delta]$ (conservation), the variational problem is formulated as

$$\sum \delta n_i \frac{\delta S[n_1, n_2 \ldots]}{\delta n_i} = 0 ,$$

$$\sum n_i = N , \qquad \sum \varepsilon_i n_i = W ,$$

where $n_1, n_2, \ldots$ are the numbers of mesons of momentum $k_1, k_2, \ldots$ respectively, and N is the total number of emitted mesons and W is the total energy.

But to see clearly the relation between the S-matrix and the most probable state we will transform the S-matrix according to the method of URSEL in the statical mechanics of imperfect gases ([17]). Putting firstly $|\langle \boldsymbol{P}_1^0 \boldsymbol{P}_2^0 | I | \boldsymbol{P}_1 \boldsymbol{P}_2 \rangle|^2 = C(\boldsymbol{P}_1 \boldsymbol{P}_2)$ we define a new function $f_1(\boldsymbol{k})$ by $|\langle \boldsymbol{P}_1^0 \boldsymbol{P}_2^0 | I | \boldsymbol{P}_1 \boldsymbol{P}_2, \boldsymbol{k} \rangle|^2 \equiv C(\boldsymbol{P}_1 \boldsymbol{P}_2) f_1(\boldsymbol{k})$; and further $f_2(\boldsymbol{k})$ is defined by $|\langle \boldsymbol{P}_1^0 \boldsymbol{P}_2^0 | I | \boldsymbol{P}_1 \boldsymbol{P}_2, \boldsymbol{k}_1 \boldsymbol{k}_2 \rangle|^2 \equiv C(\boldsymbol{P}_1 \boldsymbol{P}_2)[f_1(\boldsymbol{k}_1) f_1(\boldsymbol{k}_2) + f_2(\boldsymbol{k}_1 \boldsymbol{k}_2)]$. Repeating such procedure the S-matrix is rewritten as

$$\text{(2.4)} \quad \begin{cases} |\langle \boldsymbol{P}_1^0 \boldsymbol{P}_2^0 | I | \boldsymbol{P}_1 \boldsymbol{P}_2 \rangle|^2 = C(\boldsymbol{P}_1 \boldsymbol{P}_2) \\ |\langle \boldsymbol{P}_1^0 \boldsymbol{P}_2^0 | I | \boldsymbol{P}_1 \boldsymbol{P}_2, \boldsymbol{k} \rangle|^2 = C(\boldsymbol{P}_1 \boldsymbol{P}_2) f_1(\boldsymbol{k}) \\ |\langle \boldsymbol{P}_1^0 \boldsymbol{P}_2^0 | I | \boldsymbol{P}_1 \boldsymbol{P}_2, \boldsymbol{k}_1 \boldsymbol{k}_2 \rangle|^2 = C(\boldsymbol{P}_1 \boldsymbol{P}_2)[f_1(\boldsymbol{k}_2) f_1(\boldsymbol{k}_2) + f_2(\boldsymbol{k}_1 \boldsymbol{k}_2)] \\ |\langle \boldsymbol{P}_1^0 \boldsymbol{P}_2^0 | I | \boldsymbol{P}_1 \boldsymbol{P}_2, \boldsymbol{k}_1 \boldsymbol{k}_2 \boldsymbol{k}_3 \rangle|^2 = C(\boldsymbol{P}_1 \boldsymbol{P}_2)[f_1(\boldsymbol{k}) f_1(\boldsymbol{k}_2) f_1(\boldsymbol{k}_3) + f_2(\boldsymbol{k}_1 \boldsymbol{k}_2) f_1(\boldsymbol{k}_3) + \\ \qquad + f_1(\boldsymbol{k}_1) f_2(\boldsymbol{k}_2 \boldsymbol{k}_3) + f_1(\boldsymbol{k}_2) f_2(\boldsymbol{k}_1 \boldsymbol{k}_3) + f_3(\boldsymbol{k}_1 \boldsymbol{k}_2 \boldsymbol{k}_3)] . \end{cases}$$

([17]) See for example HILL: *Statistical Mechanics* (1956).

In this expression the S-matrix is characterized by a series of functions $f_1(k)$, $f_2(\boldsymbol{k}_1\boldsymbol{k}_2)$, ... etc. Though the general form is rather complicated (*), if $f_2(\boldsymbol{k}_1\boldsymbol{k}_2)$, $f_3(\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3)$, ... are negligible compared to $f_1(\boldsymbol{k})$ the S-matrix is factorized in respect to the variables of emitted mesons, and the problem becomes much simpler. This is the case of the LOW theory though treated rather classically. For example, in the theory of FUKUDA-TAKEDA (6) with the PS(PS) coupling, the S-matrix was factorized and their $f_1(k)$ was (see Appendix B)

$$f_1(\boldsymbol{k}) \sim \left(\frac{g}{\mu}\right)^2, \tag{2.5}$$

g, coupling costant, μ, meson mass.

From such simplified model, one can see easily that $f_1(k)$ is a quantity intimately related to the form of the meson field (meson cloud) around the nucleon.

In the case where the S-matrix is factorized there is no correlation between

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(*) Assigning the final state by the occupation numbers n_i of mesons k_i and by the « cluster » numbers $n^{(l)}_{ik\ldots l}$ of the correlated functions $f_l(k_i k_k \ldots; k_l)$, the cross section becomes

$$\sigma = \frac{2\pi}{v}\sum\prod_i \frac{1}{n_i!}\left[\frac{g^2}{\mu^2} f_1(\boldsymbol{k})\Omega_i\Delta_i\right]^{n_i} \prod_{ij}\frac{1}{n^{(2)}_{ij}}\left[\frac{1}{2!}\left(\frac{g^2}{\mu^2}\right) f_2(\boldsymbol{k}_i\boldsymbol{k}_j)\Omega_i\Omega_j\Delta_i\Delta_j\right]^{n^{(2)}_{ij}} \cdot$$

$$\cdot\ldots\cdot \prod_{ijk\ldots l}\frac{1}{n^{(l)}_{ijk\ldots l}}\left[\frac{1}{l!}\left(\frac{g^2}{\mu^2}\right)^l f_l(k_i k_j \ldots k_l)\Omega_i\Omega_j \ldots \Omega_l\Delta_i\Delta_j\ldots\Delta_l\right]^{n^{(l)}_{ijk\ldots l}} \cdot$$

$$\cdot\ldots\cdot \delta\Big(W - \sum_i \varepsilon_i n_i - \sum_{ij}(\varepsilon_i + \varepsilon_j) n^{(2)}_{ij} - \ldots\Big)\,,$$

where $\Delta_i = V\,\mathrm{d}\boldsymbol{k}_i/(2\pi)^3$, $\Omega_i = (g^2/\mu^3)(\mu/\varepsilon_i)$, V normalized volume, μ meson mass and g coupling constant. The problem is solved by a variational method, giving, for example, the « cluster » number distribution in the most probable state,

$$n^l(\boldsymbol{k}_1 \ldots \boldsymbol{k}_l) = \frac{1}{l!} f_l(\boldsymbol{k}_1 \ldots \boldsymbol{k}_l)\Omega_1\Omega_2 \ldots \Omega_l \exp\left[-\beta(\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_l)\right],$$

where the temperature $1/\beta$ of the system is determined by the conservation

$$W = -\frac{\partial}{\partial\beta}\sum_l \frac{1}{l!}\int \mathrm{d}\boldsymbol{k}_1 \ldots \mathrm{d}\boldsymbol{k}_l \Omega_1 \ldots \Omega_l f_l(\boldsymbol{k}_1 \ldots \boldsymbol{k}_l) \exp\left[-\beta(\varepsilon_1 + \ldots + \varepsilon_l)\right].$$

And the multiplicity is expressed finally as

$$\langle n\rangle = \sum_l \frac{1}{l!}\int \mathrm{d}\boldsymbol{k}_1 \ldots \mathrm{d}\boldsymbol{k}_l \Omega_1 \ldots \Omega_l f_l(\boldsymbol{k}_1 \ldots \boldsymbol{k}_l) \exp\left[-\beta(\varepsilon_1 + \ldots + \varepsilon_l)\right].$$

the created mesons, and the mode of production may be something like a jet of an ideal gas. In fact, as will be described later, FERMI theory just corresponds to an evaporation of this ideal meson gas from the « hot spot » of the colliding region. On the other hand, where the functions $f_2(\boldsymbol{k}_1\boldsymbol{k}_2)$, $f_3(\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3)$ etc. are not negligible there are correlations between created mesons or mesons and nucleons. In such cases the production may resemble a jet of an imperfect gas, and if the meson-meson interaction is very strong the meson cloud may behave like a fluid.

3. – Perfect gas model (Fermi type theory).

In this case, the S-matrix being factorized, the total cross section takes a simple form

$$\sigma_n = \frac{(2\pi)^4}{v_r}\int c_1(\boldsymbol{p}_1\boldsymbol{p}_2)\,\frac{\mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2}{E_1E_2}\,\frac{f_1(\boldsymbol{k}_1)f_1(\boldsymbol{k}_2)\ldots f_1(\boldsymbol{k}_n)}{n!}\cdot \tag{3.1}$$

$$\cdot\,\frac{\mathrm{d}\boldsymbol{k}_1}{\varepsilon_1}\,\frac{\mathrm{d}\boldsymbol{k}_2}{\varepsilon_2}\ldots\frac{\mathrm{d}\boldsymbol{k}_n}{\varepsilon_n}\,\delta^4(\Delta p - k_1 - k_2 - \ldots - k_n)\ ,$$

and in the occupation number representation it becomes

$$\sigma = \sum\sigma_n = \frac{(2\pi)^4}{v_r}\sum c(\boldsymbol{p}_1\boldsymbol{p}_2)\,\frac{\mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2}{E_1E_2}\prod_s\sum_{n_s}\frac{1}{n_s!}\left[f_1(k_s)\,\frac{\Delta k_s}{\varepsilon_s}\right]^{n_s}\delta^4(\Delta p - \sum n_s k_s)\ ,$$

$$\Delta k_s \equiv \frac{V}{(2\pi)^3}\,\mathrm{d}\boldsymbol{k}_s\ . \tag{3.2}$$

Then, the most probable final state is given, as was described above, by a solution of the variational problem,

$$\begin{cases} \sum\delta n_s\,\dfrac{\partial}{\partial n_s}\ln\left[\dfrac{f_1(\boldsymbol{k}_s)\,\Delta\boldsymbol{k}_s}{n_s!\,\varepsilon_s}\right]^{n_s} = 0\ , \\ \sum\delta n_s k_s = 0\ . \end{cases} \tag{3.3}$$

When n_s is sufficiently large, n_s is reduced to

$$n_s = f_1(\boldsymbol{k}_s)\,\frac{1}{\exp\left[\beta_\mu k_\mu\right]-1}\,\frac{\Delta\boldsymbol{k}_s}{\varepsilon_s}\ , \tag{3.4}$$

where the unknown constants (Lagrange multiplier) β_μ are determined by the conservation relations

$$(\Delta p)_\mu = \sum(k_s)_\mu n_s\ . \tag{3.5}$$

Or, in momentum representation

$$n(\boldsymbol{k}) = \frac{f_1(\boldsymbol{k})}{\exp[\beta_\mu k_\mu] - 1} .$$

This is our starting general equation obtained by a statistical approximation of the field. The detailed comparison with the experiments by this equation will be given in Sect. 7. Here, the special form of $f_1(k)$ is discussed concerning the physical meaning of the different statistical theories of production.

To simplify the problem, neglecting the momentum conservation the above results are expressed as

$$\begin{cases} n(\boldsymbol{k}) = f_1(\boldsymbol{k}) \exp[-\beta\varepsilon_k] \dfrac{\mathrm{d}\boldsymbol{k}}{\varepsilon_k} , \\ \Delta W = \int f_1(\boldsymbol{k}) \exp[-\beta\varepsilon_k] \mathrm{d}\boldsymbol{k} , \end{cases} \tag{3.6}$$

where ΔW is a difference of energy between the initial and final states of the nucleons. If we, as LOW, take for the meson field the form $f_1(k) = g^2/\mu^2$, the distribution $n(k)$ becomes,

$$\begin{cases} n(\boldsymbol{k}) = \dfrac{g^2}{\mu^3} \sqrt{1 - v_k^2} \exp[-\beta\varepsilon_k] \mathrm{d}\boldsymbol{k} , \\ v_k \equiv \dfrac{|\boldsymbol{k}|}{\varepsilon} . \end{cases} \tag{3.7}$$

This is just the result of LOW. On the other hand, if the eigen field function $f_1(k)$ is taken as a contracted volume of FERMI

$$f_1(\boldsymbol{k}) = \frac{2M}{E_0} \Omega_0 \varepsilon_k , \tag{3.8}$$

we get justly the result of FERMI.

$$\begin{cases} n(\boldsymbol{k}) = \Omega_0 \sqrt{1 - v_0^2} \exp[-\beta\varepsilon_k] \mathrm{d}\boldsymbol{k} , \\ \sqrt{1 - v_0^2} \equiv \dfrac{2M}{E_0} . \end{cases} \tag{3.9}$$

Now we will remember the meaning of the Fermi theory from a general point of view. The Fermi theory is based on the intuitive supposition that from a hot spot of thermoequilibrium state in the collision region mesons are emitted as by an evaporation. Owing to the high velocity of the incident nucleons, the contracted volume of the hot spot is

$$\Omega_F = \Omega_0 \sqrt{1 - v_0^2} \sim \sqrt{1 - v_0^2} \frac{4}{3} \pi \left(\frac{1}{\mu}\right)^3 . \tag{3.10}$$

where v_0 is the nucleon velocity. With a thermoequilibrium hypothesis, the energy distribution becomes Planck's one

$$n(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k} = \frac{\Omega_F\,\mathrm{d}\boldsymbol{k}}{\exp\left[\beta\varepsilon_k\right] - 1}\,, \tag{3.11}$$

where the temperature $1/\beta$ of the system is determined by the conservation of energy.

$$\int \varepsilon_k n(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k} = E_0\,, \tag{3.12}$$

where E_0 is the incident kinetic energy of the nucleons in C.M. system. Then, the multiplicity is given by

$$\langle n\rangle = \sim \left(\frac{E_0}{M}\right)^{\frac{1}{2}} \sim \left(\frac{E_{\mathrm{lab}}}{M}\right)^{\frac{1}{4}}, \tag{3.13}$$

in good accordance with the experiments.

As was seen above (Eq. (3.9)), FERMI results are obtained also by an approximate method in the field theory. It should be noted however, that what was derived are the results of FERMI, but not his picture of thermoequilibrium state. In a field theoretical way, the problem of meson multiple production is to calculate any probability

$$P \equiv P(n_1 n_2 \ldots;)$$

for production of n_1 mesons of wave number k_1, n_2 mesons of wave number k_2 and so on. In practice however, it is too complicated because of the large number of produced mesons, and even a calculation of the phase volume is almost impossible. But for an exceptional process one can calculate the probability. This is the case where the eigenfield of nucleons is shaked off at random by the shock of collision, and the probability becomes, as is easily seen ([19]), of the Poisson type,

$$Pp \sim \frac{f(\boldsymbol{k}_1)^{n_{k_1}}}{n_{k_1}!}\,\frac{f(\boldsymbol{k}_2)^{n_{k_2}}}{n_{k_2}!} \ldots\,, \tag{3.14}$$

$f(k)$ being a probability of finding k mesons in the eigenfield. This form of $f(k)$ is, of course, dependent on the Lorentz contraction of the eigenfield. In general, the process of shaking off k_1 mesons after shaking off k_2 mesons

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([18]) F. BLOCH and A. NORDSIECK: *Phys. Rev.*, **52**, 54 (1937).
([19]) W. PAULI and M. FIERZ: *Nuovo Cimento*, **15**, 1 (1938).

is different from the shaking off in a inverse way, so the supposition « at random » is not exact. Even so, this « at random » may be a rather good approximation and forms the standing point of LOW. In such way, the $f(k)$ was calculated and used by LOW in a form $f(\boldsymbol{k}) \sim (g^2/\mu^3)(\mu/\varepsilon_k)$.

Accordingly noting the similar form and same order of two factors, $(g^2/\mu^3)\sqrt{1-v_k^2}$ in LOW and $(4/3)\pi(1/\mu^3)\sqrt{1-v_0^2}$ in FERMI (eqs. (3.7) and (3.9)), FERMI theory may be considered as an intuitive one for the statistical treatment of the field theory. But our most probable state method (hereafter called as Fermi-type theory) contains, in the physical contents, much more than the thermoequilibrium state, and is more rich in the physical interpretations. This point will be seen in detail in Sect. 7. An example to clarify the meaning of Fermi theory more concretely is given in Appendix B.

4. – Imperfect meson gas.

In Sect. 2 the structure of the S-matrix was discussed by the Ursel operation, and in Sect. 3 the special cases were considered where the S-matrix is factorizable. When the S-matrix cannot be factorized due to the strong correlation between emitted mesons, one may treat it as an imperfect gas. But though, as will be seen in Sect. 7, the perfect gas model (Fermi-type theory) is sufficiently good, at least in the present stage, to explain the recent experimental data, we shall give here an imperfect gas model mathematically and abstractly.

For this purpose a functional $\sigma[Z]$ is introduced as follows, where $Z(k)$ is an arbitrary parametric function

$$\sigma[Z] = \sum \sigma_n[Z] = \frac{(2\pi)^4}{v_r}\int \frac{\mathrm{d}\boldsymbol{p}_1}{E_1}\frac{\mathrm{d}\boldsymbol{p}_2}{E_2}\sum_n\sum_{i=1}^{n}\int \frac{\mathrm{d}\boldsymbol{k}_i}{\varepsilon_i} Z(\boldsymbol{k}_i)\cdot \cdot |\langle \boldsymbol{p}_1^0\boldsymbol{p}_2^0|I|\boldsymbol{p}_1\boldsymbol{p}_2;\ \boldsymbol{k}_1 \dots \boldsymbol{k}_n\rangle|^2\,\delta^4(\Delta p - \sum k_i)\,. \tag{4.1}$$

As will be proved easily, this functional satisfies the following relations.

1) $$\sigma[Z=1] = \sum \sigma_n = \sigma_{\mathrm{tot}}\,.$$

2) $$\frac{n!}{2\pi i}\oint \frac{\sigma[Z=\zeta]}{\zeta^{n+1}}\,\mathrm{d}\zeta = \sigma_n\,.$$

3) $$\left.\frac{\delta\sigma[Z]}{\delta Z(\boldsymbol{k})}\right|_{Z=1} \equiv \sigma(\boldsymbol{k}),$$

where $\sigma(k)$ is the cross section of production of the k-th meson.

4) $$\frac{\delta^2\sigma[Z]}{\delta Z(\boldsymbol{k})\,\delta Z(\boldsymbol{k}')} = \sigma(\boldsymbol{k}.\ \boldsymbol{k}'),$$

where $\sigma(kk')$ is the cross section of production of the k-th and k'-th mesons. Or, setting $\sigma(Z) \equiv \exp[S(Z)]$ one obtains

$$(4.3)\qquad \begin{cases} \left.\dfrac{\delta S[Z]}{\delta Z(\boldsymbol{k})}\right|_{Z=1} = \dfrac{\sigma(\boldsymbol{k})}{\sigma_{\mathrm{tot}}}\,, \\ \left(\dfrac{\delta S(Z)}{\delta Z(\boldsymbol{k})}\,\dfrac{\delta S(Z)}{\delta Z(\boldsymbol{k}')} + \dfrac{\delta^2 S(Z)}{\delta Z(\boldsymbol{k})\,\delta Z(\boldsymbol{k}')}\right)_{Z=1} = \dfrac{\sigma(\boldsymbol{k}\boldsymbol{k}')}{\sigma_{\mathrm{tot}}}\,. \end{cases}$$

Introducing (2.4) into (4.1),

$$(4.4)\qquad \sigma_n[Z] = \frac{(2\pi)^4}{v_r}\int \frac{\mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2}{E_1E_2}\,c(\boldsymbol{p}_1\boldsymbol{p}_2)\prod_l \sum_{\substack{m_l=0 \\ (\sum_l m_l l = n)}} \frac{1}{(l\,!)^{m_l} m_l\,!}\cdot$$

$$\cdot\left[\int \prod_{i=1}^{l} \frac{\mathrm{d}\boldsymbol{k}_i}{\varepsilon_i}\,Z(\boldsymbol{k}_i) f_l(\boldsymbol{k}_1 \ldots \boldsymbol{k}_l)\right]^{m_l} \delta^4(\Delta p - \sum k)\,.$$

Remembering that the δ function is $\delta^4(x) = (1/(2\pi)^4)\int \exp[-i\xi\cdot x]\mathrm{d}\xi$, $\sigma[Z]$ is expressed finally

$$\sigma[Z] = \sum_n \sigma_n[Z] = \int \frac{\mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2}{v_r E_1 E_2}\,c(\boldsymbol{p}_1\boldsymbol{p}_2)\int \mathrm{d}^4\xi\,\exp\left[-i\Delta p\xi + \sum_l b_l(Z,\xi)\right],$$

where

$$(4.5)\qquad b_l[Z,\xi] = \frac{1}{l\,!}\int \frac{\mathrm{d}\boldsymbol{k}_1}{\varepsilon_1}\ldots\frac{\mathrm{d}\boldsymbol{k}_l}{\varepsilon_l}\,f_l(\boldsymbol{k}_1\ldots\boldsymbol{k}_l)\exp\left[i(k_1+\ldots+k_l)\cdot\xi\right]\cdot Z(\boldsymbol{k}_1)\ldots\cdot Z(\boldsymbol{k}_l)\,.$$

Then, the most probable state of $\sigma[Z]$ is evaluated by the saddle point method and given

$$(4.6)\qquad \sigma[Z] \cong \int \frac{\mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2}{v_r E_1 E_2}\,c(\boldsymbol{p}_1\boldsymbol{p}_2)\exp\left[-\Delta p\cdot\beta + \sum_l b_l(Z,\xi = -i\beta)\right],$$

where β is determined by the conservation relation

$$(4.7)\qquad \Delta p = \sum_l \frac{1}{l\,!}\int \frac{\mathrm{d}\boldsymbol{k}_1}{\varepsilon_1}\ldots\frac{\mathrm{d}\boldsymbol{k}_l}{\varepsilon_2}\,f_l(\boldsymbol{k}_1\ldots\boldsymbol{k}_l)\exp\left[\beta\cdot(k_1+\ldots+k_l)\right].$$

In such formulations, the branching ratio of production becomes

$$(4.8)\qquad \begin{cases} \dfrac{\sigma(\boldsymbol{k})}{\sigma_{\mathrm{tot}}} = \displaystyle\sum_l \left.\frac{\delta b_l[Z]}{\delta Z(\boldsymbol{k})}\right|_{Z=1}, \\ \qquad\qquad = \displaystyle\sum_l \frac{1}{(l-1)\,!}\int f_l(\boldsymbol{k}\boldsymbol{k}_2\ldots\boldsymbol{k}_l)\exp\left[\beta(k+k_2+\ldots k_l)\right]\frac{\mathrm{d}\boldsymbol{k}_l}{\varepsilon_2}\ldots\frac{\mathrm{d}\boldsymbol{k}_l}{\varepsilon_l}\,. \end{cases}$$

Consequently, if $\sigma(\boldsymbol{k})$, $\sigma(\boldsymbol{k}, \boldsymbol{k}')$, ... etc. are obtained experimentally one may get a good information on the S-matrix structure, that is, the structure of the meson field.

5. – Hydrodynamical behavior of the meson field.

The fact that Fermi theory could not explain the distributions of angles and momenta in an air shower might show an incorrecteness of neglecting the correlations between the created mesons. Then, LANDAU gave a reasonable interpretation of these phenomena supposing that the once created mesons in the mode of FERMI are emitted like a jet of fluid, in place of gas evaporation, because of the strong interaction and radiation pressure between mesons in the contracted volume. But his arguments were always classical (not quantum-mechanical) though very relativistic. So, from a field theoretical consideration we shall derive some hydrodynamical features of the meson field which were the starting point of the Landau theory.

For simplicity, we consider an ensemble of the charged free mesons whose momenta are $k_1, k_2, \ldots k_n$. In such state $\Psi(\boldsymbol{k}_1 \ldots \boldsymbol{k}_n)$ the following quantities are examined.

$$\varrho(x) \equiv \langle \Psi^*, \Phi^*(x)\Phi(x)\Psi \rangle \,, \tag{5.1}$$

$$J_\mu(x) \equiv \left\langle \Psi^*, \Phi^*(x) \frac{\vec{\partial}_\mu - \overleftarrow{\partial}_\mu}{2i} \Phi(x)\Psi \right\rangle \,, \tag{5.2}$$

$$T_{\mu\nu}(x) \equiv \left\langle \Psi^*, \Phi^*(x) \frac{\vec{\partial}_\mu - \overleftarrow{\partial}_\mu}{2i} \frac{\vec{\partial}_\nu - \overleftarrow{\partial}_\nu}{2i} \Phi(x)\Psi \right\rangle \,. \tag{5.3}$$

The meaning of these quanties is obvious, $\varrho(x)$ density, $J_\mu(x)$ current and $T_{\mu\nu}(x)$ energy momentum tensor. By the Fourier transformation

$$\Phi(x) = \int \Phi(k) \exp\left[ikx\right] \mathrm{d}k \,, \tag{5.4}$$

they are rewritten

$$\varrho(x) = \int \mathrm{d}k \, \mathrm{d}k' \exp\left[-i(k-k')x\right] \varrho(kk') \,, \tag{5.1'}$$

$$J_\mu(x) = \int \mathrm{d}k \, \mathrm{d}k' \exp\left[-i(k-k')x\right] \left(\frac{k+k'}{2}\right)_\mu \varrho(kk') \,, \tag{5.2'}$$

$$T_{\mu\nu}(x) = \int \mathrm{d}k \, \mathrm{d}k' \exp\left[-i(k-k')x\right] \left(\frac{k+k'}{2}\right)_\mu \left(\frac{k+k'}{2}\right)_\nu \varrho(kk') \,, \tag{5.3'}$$

where

$$\varrho(kk') \equiv \langle \Psi, \Phi^*(k)\Phi(k')\Psi \rangle \,. \tag{5.5}$$

Or, for the physical interpretation, it will be useful to transform them as following

$$\varrho(x) = \int \mathrm{d}p \varrho(p, x) \,, \tag{5.1''}$$

$$J_\mu(x) = \int \mathrm{d}p p_\mu \varrho(p, x) \,, \tag{5.2''}$$

$$T_{\mu\nu}(x) = \int \mathrm{d}p p_\mu p_\nu \varrho(p, x) \,, \tag{5.3''}$$

where

$$\left\{ \begin{array}{l} \dfrac{k+k'}{2} \equiv p \,, \qquad k - k' \equiv l \,, \qquad \varrho(kk') \equiv \varrho(p, l) \,, \\ \varrho(p, x) \equiv \displaystyle\int \varrho(p, l) \exp\left[ilx\right] \mathrm{d}l \,. \end{array} \right. \tag{5.6}$$

As Ψ contains only the free mesons, the matrix elements $\varrho(kk') \equiv \langle k \,|\, \varrho \,|\, k' \rangle$ have no vanishing values only when $k^2 + \mu^2 = k'^2 + \mu^2 = 0$, where μ is the meson mass. Then, $J_\mu(x)$ and $T_{\mu\nu}(x)$ are proved to satisfy the continuity equation

$$\frac{\partial}{\partial x_\mu} J_\mu(x) = 0 \,, \tag{5.7}$$

$$\frac{\partial}{\partial x_\nu} T_{\mu\nu}(x) = 0 \,, \tag{5.8}$$

and also it is seen immediately

$$\sum_\mu T_{\mu\nu}(x) = -\mu^2 \langle \Psi^*, \Phi^*(x)\Phi(x)\Psi \rangle \tag{5.9}$$

And, $J_\mu(x)$ being the current density, the quantities u_μ, defined by

$$u_\mu(x) = \frac{J_\mu(x)}{\sqrt{-J_\mu(x)J_\mu(x)}} \,, \tag{5.10}$$

correspond to the velocity of fluid which fulfils the 4-dimensional velocity relation $-u_\mu u_\mu = 1$.

As is known in the fluidmechanics the average value of particle velocities k_i/ε represents the fluid velocity, and the mean square of the difference of fluid velocity and particle velocity gives rise to the pressure or viscosity. To understand these characters it is convenient to transit to the coordinate system where the fluid velocity is zero, that is, in this new coordinate the velocity vector becomes

$$\bar{u}_\mu(x) = (0, 0, 0, 1) . \tag{5.11}$$

This Lorentz trasformation is represented by $a_{\mu\nu}$,

$$\begin{cases} u_\mu(x) = a_{\mu\nu}(x)\bar{u}_\mu(\bar{x}) = a_{\mu\nu} \\ k_\mu = a_{\mu\nu}\bar{k}_\nu \\ T_{\mu\nu}(x) = a_{\mu\mu'}\bar{T}_{\mu'\nu'}(\bar{x})a_{\nu'\nu} \end{cases} \tag{5.12}$$

Now, we suppose here that in the coordinate system flowing with the meson fluid the distribution of momentum of composite mesons should be isotropic. For a free field this may be a rather natural assumption. Then, $k+k'/2=p$ is also isotropic in this system, and the energy momentum tensor $T_{\mu\nu}$ is transformed as following

$$\begin{cases} T_{\mu\nu}(x) = a_{\mu\mu'}\bar{T}_{\mu'\nu'}(\bar{x})a_{\nu'\nu} , \\ \qquad = a_{\mu\mu'}\int \mathrm{d}\bar{p}\,\bar{p}_{\mu'}\bar{p}_{\nu'}\varrho(\bar{p}, \bar{x})a_{\nu'\nu} , \\ \qquad = \sum_{i=1}^{3} a_{\mu i}a_{\nu i}\,\frac{1}{3}\int \mathrm{d}\bar{p}\bar{\boldsymbol{p}}^2\varrho(\bar{p}\bar{x}) + a_{\mu 0}a_{\nu 0}\int \mathrm{d}\bar{p}\,\bar{p}_0^2\varrho(\bar{p}\bar{x}) . \end{cases} \tag{5.13}$$

By introducing the two quantities $p(x)$ and $\varepsilon(x)$ (*)

$$\begin{cases} \dfrac{1}{3}\int \mathrm{d}p\boldsymbol{p}^2\varrho(p, x) \equiv p(x), \\ \int \mathrm{d}p\,\bar{p}_0^2\varrho(p, x) \equiv \varepsilon(x) . \end{cases} \tag{5.14}$$

(*) In the case where the momentum distribution of mesons is not isotropic, the definitions of $p(x)$ and $\varepsilon(x)$ should be taken as

$$p(x) \equiv \tfrac{1}{3}\sum^{3}\bar{T}_{ii}(\bar{x}) ,$$

$$\varepsilon(x) = \bar{T}_{00}(\bar{x}).$$

$T_{\mu\nu}$ takes finally a familiar form

$$\left\{\begin{aligned} T_{\mu\nu}(x) &= p(x)\sum_{i=1}^{3} a_{\mu i}a_{\nu i} + \varepsilon(x)u_{\mu}(x)u_{\nu}(x) \\ &= p(x)(\delta_{\mu\nu} + u_{\mu}u_{\nu}) + \varepsilon(x)u_{\mu}u_{\nu}\,, \end{aligned}\right. \tag{5.15}$$

where the orthogonality condition of Lorentz transformation is used.

As is readily seen from the defining equations, $p(x)$ and $\varepsilon(x)$ represent the pressure density and energy density respectively. Consequently, this $T_{\mu\nu}$ is just the energy momentum tensor of a perfect liquid proposed first by LANDAU.

Between the pressure and energy there exists a relation

$$p(x) \leqslant \tfrac{1}{3}\varepsilon(x)\,, \tag{5.16}$$

because of

$$\left\{\begin{aligned} \sum_{\mu} T_{\mu\nu}(x) &= p(x)(4 + \sum u_{\mu}^{2}) + \varepsilon(x)\sum_{\mu} u_{\mu}^{2} = 3p(x) - \varepsilon(x)\,, \\ &= -\mu^{2}\langle \Psi\Phi^{*}\Phi\Psi\rangle \leqslant 0\,. \end{aligned}\right. \tag{5.17}$$

In the limiting case $p(x) = \varepsilon(x)/3$ which is called STEFAN-BOLTZMANN relation realized in the thermoequilibrium state, this is just the state equation of LANDAU. Therefore, one sees that an ensemble of free mesons behaves, to a certain degree, as like a fluid obeying the hydrodynamical relations of LANDAU.

6. – Comments on Landau's theory.

LANDAU considered the multiple production process in three steps: 1) The formation of local thermoequilibrium in a contracted meson cloud because of the strong interaction and high density of mesons. 2) A hydrodynamical linear expansion of the meson cloud, due to the radiation pressure, in the direction normal to the contracted disk. 3) The three dimentional emission of free particles owing to the thin density of nuclear materials. Such description was proved to be very effective to explain the angular distribution of air showers that the Fermi theory failed to do.

As was discussed in the previous section the meson field is described as a fluid if the meson velocity distribution is isotropic with respect to the fluid's velocity. In the Landau theory where the particles arrive at the state of thermoequilibrium as in the Fermi theory this condition of isotropy is satisfied. However it may appear that this isotropy should contradict the linear expansion of the meson fluid of LANDAU. In fact however, the equilibrium region being contracted like a disk the pressure has a large gradient in the normal direc-

tion, say x, of the disk, and it is this gradient that forces the linear expansion of the Landau fluid. The pressure is given by (see Eq. (5.14)),

$$p(x) = \frac{1}{3}\int \mathrm{d}p\, \bar{\boldsymbol{p}}^2 \varrho(p, x) = \frac{1}{3}\int \mathrm{d}p \cdot \boldsymbol{p}^2 \int \mathrm{d}l \exp\left[il \cdot x\right] \varrho(p, l) \,. \tag{6.1}$$

On the other hand, Fermi's equilibrium volume will expand isotropically with the thermal velocity

$$v_{\mathrm{th}} = \left[\int \mathrm{d}\boldsymbol{p} \left(\frac{\boldsymbol{p}}{p_0}\right)^2 \varrho(p, x) \Big/ \int \varrho(p, x)\, \mathrm{d}p\right]^{\frac{1}{2}}. \tag{6.2}$$

Then, in order to realize Landau's linear expansion the spurting velocity $|l| \sim lx$ must be much larger than the thermal velocity v_{th}. By definition of p and l we have a relation for the momentum of emitted mesons

$$\boldsymbol{k} = \boldsymbol{p} \pm \frac{\boldsymbol{l}}{2} \,. \tag{6.3}$$

So, the above condition of linear expansion becomes

$$\langle \boldsymbol{k}_x^2 \rangle = \langle \boldsymbol{p}_x^2 \rangle + \frac{\langle l_x^2 \rangle}{4} \gg \langle \boldsymbol{k}_y^2 \rangle, \langle \boldsymbol{k}_y^2 \rangle \,, \tag{6.4}$$

namely, at the instant of creation the x component of meson momentum must have values very much larger than the other components.

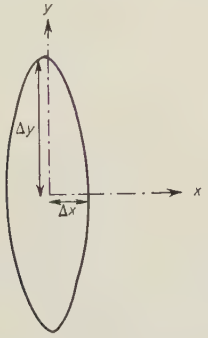

Fig. 1.

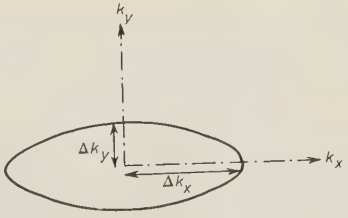

Fig. 2.

Accordingly, in order to understand Landau's hydrodynamical theory as an approximation of the field theory in the high energy region, the momentum distribution of mesons at the initial time of creation should be pictured as in Fig. 1. Or, in the configuration space, this may correspond to the contraction of the meson cloud as in Fig. 2, where the uncertainty relation $\langle \Delta x \rangle \langle \Delta k_x \rangle \sim 1$ exists. If

the multiple production process is a shaking off of the contracted proper field (as in Fig. 2), the initial condition of momentum (as in Fig. 1) should be rather reasonable. In fact, as will be seen in the next section, only by this initial condition of momentum one can obtain almost all the same results as that of LANDAU. Namely, in his theory only the initial condition is of vital importance, and the complicated hydrodynamical treatment may not be essential. In other words, if only his initial condition is taken one can obtain equally good results by a more familar and usual method. Then, the success of Laudau's theory does not necessarily mean that the meson field is degenerate like a fluid in the extremely high energy region.

According to KOBA ([3]), in the Landau theory the essential features of the extremely high energy events are determined principally by the low energy processes in these phenomena. If so, the ultra high energy events may be not so valuable, in spite of our expectation, to obtain useful information on the interactions of elementary particles in the high energy region. However, the bases of the hypothesis are less clear and one should be more careful to give them up, and it should be noted that the characters of the Fermi theory have not yet been clarified. The experimental basis of the conclusion that the ultra high energy characters are rather dependent on the low energy process is the branching ratio of θ^0 and π^0 production. Being impossible to explain it by the Fermi theory, KOBA had recourse to the strong final interaction of the Landau theory. But there exists also another interpretation of this branching ratio basing only on a creation mechanism in high energy processes without any strong final interaction (fluid approximation) in low energy processes. In fact, recurring to the Fermi theory as an approximation of the field theory (Fermi type theory), we shall show in the next section that one can get, in a rather general way, almost all the same results as those of LANDAU.

7. – A simpler model (Fermi type theory)

FERMI derived his results basing on the picture that in a small contracted volume of meson cloud the thermoequilibrium states are attained immediately after the high energy collision of nucleons. Then, the velocity distribution of particles in equilibrium state is necessarily isotropic, and in order to explain the sharply forward collimated angular distribution he had to resort to the restriction of phase volume by the conservation of angular momentum in collision. In this case the details of the interaction matrix elements are of no more importance.

As described roughly in Sect. 3 a theory very similar to Fermi's but much richer in physical contents was obtained by treating the field theory with

analogy to the statistical mechanics. Namely, in the case that many mesons are produced, ignoring the detailed knowledge of the probability one has only to consider the most probable state attained by the produced particles.

In the simplest case where the meson fields are shaked off at random by a shock of collision the most probable distribution becomes

$$n(\boldsymbol{k}) = \frac{f(\boldsymbol{k})}{\exp[\beta\varepsilon_k] - 1}, \tag{7.1}$$

depending on the functions $f(k)$ related to the form of the eigenfield of the nucleons. As was seen in the case of LOW these $f(k)$ belong to the spectrum of the eigenfield of the nucleons. But we have no detailed knowledge of the eigenfield of the nucleons, especially in the extremely high energy regions. Rather, we should obtain some information on the eigenfield by the analysis of the high energy events. Then, we shall suppose only the following qualitative characters of the eigenfield.

1) The extension of the eigenfield of a rest nucleon is of the order of the Compton wave length of the meson, that is

$$\langle \Delta x \rangle = \langle \Delta y \rangle = \langle \Delta z \rangle \sim \frac{\hbar}{\mu c}. \tag{7.2}$$

Then, for a nucleon moving in the X-direction, (see Fig. 2)

$$\left| \begin{array}{l} \langle \Delta x \rangle \sim \left(\dfrac{E_0}{M}\right)\dfrac{1}{\mu}, \\ \langle \Delta y \rangle = \langle \Delta z \rangle \sim \dfrac{1}{\mu}, \end{array} \right. \qquad (\hbar = c = 1), \tag{7.3}$$

where E_0 means the energy of colliding nucleons in the center of mass system.

2) In the collision of nucleons the mesons are produced as shaking off of the eigenfield. For the wave-packet of mesons there exist the uncertainty relations (see Fig. 1)

$$\langle \Delta x \rangle \langle \Delta k_x \rangle \sim 1. \tag{7.4}$$

3) Though our distribution $n(\boldsymbol{k})$ contains the eigenfield spectrum $f(k)$, the multiplicity predicted coincides with that of FERMI

$$\langle n_\pi \rangle = \nu \left(\frac{E_0}{M}\right)^{\frac{1}{4}}, \tag{7.5}$$

ν, costant.

Such hypotheses appear to restrict too much the spectrum $f(k)$, especially the hypothesis 3) might seem to be impossible. In fact however, these assump-

tions do not impose severe restrictions, and as will be shown in Appendix A a simple step function

$$(7.6)\qquad \begin{cases} f(k_x k_y k_z) = 1 & \text{if} \begin{cases} -(E_0/M)\mu < k_x < (E_0/M)\,, \\ -\mu < k_y < \mu\,, \\ -\mu < k_z < \mu\,, \end{cases} \\ = 0 & \text{otherwise} \end{cases}$$

satisfies these conditions. The physical picture of this function $f(k)$ is evident.

In the original Fermi theory $f(k)$ comes from the matrix element and is considered as a constant factor which plays a role of contraction of the equilibrium volume. In our case this $f(k)$ is responsible for the angular distribution of emitted particles. FERMI supposes a realization of thermoequilibrium state, while we think the production process as a shaking off of the eigenfield in the form of wave packet. This difference appears also in the branching ratio θ^0 and π^0 production.

By the hypothesis 3) the meson kinetic energy per meson emitted, i.e., the temperature $1/\beta$ is given by the conservation of energy.

$$(7.7)\qquad \frac{1}{\beta} = \frac{2E_0}{\langle n_\pi \rangle} = \frac{2E_0}{\nu (E_0/M)^{\frac{1}{2}}} = \frac{2}{\nu}\left(\frac{E_0}{M}\right)^{\frac{1}{2}} M\,,$$

where ν is an unknown constant. But from the case of Shein's star (20 emitted mesons at a laboratory energy of 10^{13} eV), one can obtain $\nu = 2.4$. One sees that $1/\beta$ is sufficiently larger than the pion mass μ. Then, the average value of lateral component of meson momentum can be approximated as

$$(7.8)\qquad \langle k_\perp \rangle = \frac{\int |k_y|\, n(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}}{\int n(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}} \sim \frac{\int |k_y|\, f(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}}{\int f(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}}\,,$$

because k_y integral is now mostly dependent on $f(k)$. Consequently,

$$(7.9)\qquad \langle k_\perp \rangle \sim \langle \Delta k_y \rangle \sim \mu = 140\ \mathrm{MeV/c}\,,$$

which just explains the experimental results of NISHIMURA for the lateral component of meson momentum.

Next let us consider the angular distribution of emitted particles. In our method the averaged angle that the particles make with the air shower axis is given simply (in C.M. system)

$$(7.10)\qquad \theta^{\mathrm{C.M.}} = \frac{\langle k_\perp \rangle}{\langle k_x \rangle} \sim \frac{\mu}{\langle k_x \rangle}\,.$$

On the other hand, due to our assumptions 1) and 2) $k_x \gg k_y = k_z$ (this is just the initial condition of LANDAU), k_x is estimated as

$$\langle k_x \rangle = \frac{\int k_x n(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}}{\int n(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}} \sim \frac{\int \varepsilon_x n(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}}{\langle n_\pi \rangle} \sim \frac{2E_0}{n_\pi} = \frac{2}{\nu}\left(\frac{E_0}{M}\right)^{\frac{1}{2}} M\,. \tag{7.11}$$

Accordingly, the average angle in the C.M. system becomes

$$\langle \theta^{\mathrm{C.M.}} \rangle = \frac{\nu\mu}{2M}\left(\frac{E_0}{M}\right)^{\frac{1}{2}}.$$

Transformed to the laboratory system this angle is expressed as

$$\langle \theta^{\mathrm{Lab}} \rangle = \frac{\nu\mu}{4M}\left(\frac{M}{E_0}\right)^{\frac{3}{2}} = \frac{\nu\mu}{4M}\left(\frac{2M}{E_{\mathrm{Lab}}}\right)^{\frac{3}{4}},$$

where the relations

$$\frac{E_0}{M} \sim \left(\frac{E_{\mathrm{Lab}}}{2M}\right)^{\frac{1}{2}}, \qquad \theta^{\mathrm{Lab}} \sim \left(\frac{M}{E_0}\right)\frac{\theta^{\mathrm{C.M.}}}{2}\,,$$

are used.

Now taking the empirical value $\nu = 2.4$ based on Shein's star, the final expression for θ^{Lab} is

$$\langle \theta^{\mathrm{Lab}} \rangle \sim 0.15 \cdot \left(\frac{M}{E_{\mathrm{Lab}}}\right)^{\frac{3}{4}}.$$

For Fermi's and Laudau's theories, TCHELFSKY and ROSENTHAL had given the half angle $\theta_{\frac{1}{2}}$ in which one half of the energy is emitted, by calculating the energy flow εu_μ (ε is the energy density, u_μ the velocity). Our results are compared, in Table I, with those given by them. As is seen, our results are essentially the same as that of LANDAU.

TABLE I.

E_{Lab}	10^{13} eV	10^{14} eV	10^{15} eV
θ^{Lab} (Ours)	$1.5 \cdot 10^{-4}$	$2.7 \cdot 10^{-5}$	$4.8 \cdot 10^{-6}$
$\theta_{\frac{1}{2}}$ (LANDAU)	$1.3 \cdot 10^{-4}$	$2.2 \cdot 10^{-5}$	$3.8 \cdot 10^{-6}$
$\theta_{\frac{1}{2}}$ (FERMI)	$6.7 \cdot 10^{-4}$	$2.1 \cdot 10^{-4}$	$6.7 \cdot 10^{-5}$

Finally we consider the branching ratio for θ^0 and π^0 production. As the simplest example, we take the spectrum $f(k)$ as that of LOW, i.e., $f(\boldsymbol{k}) = (g^2/\mu^3)(\mu/\varepsilon_k)$. Then the multiplicity of π^0 becomes, as was described in Sect. 3,

$$\langle n_{\pi^0} \rangle = \frac{g_\pi^2}{\mu_\pi^2}\int \frac{1}{\exp\left[\beta\varepsilon_\pi\right] - 1}\,\frac{\mathrm{d}\boldsymbol{k}}{\varepsilon_\pi}\,, \tag{7.16}$$

and in the same way for θ^0 mesons,

$$\langle n_{\theta^0} \rangle = \frac{g_\theta^2}{\mu_\theta^2} \int \frac{1}{\exp[\beta \varepsilon_\theta] - 1} \frac{\mathrm{d}\boldsymbol{k}}{\varepsilon_\theta}, \tag{7.17}$$

where g_θ, g_π are coupling constants, μ_θ, μ_π the masses of θ^0 and π^0 mesons respectively. The common temperature is determined, as previously written, by the conservation of energy. This temperature being very high the integral part of the multiplicity equations are almost all the same for θ^0 and π^0, and the branching ratio is given

$$\frac{\langle n_\theta \rangle}{\langle n_\pi \rangle} \sim 2 \left(\frac{\mu_\pi}{\mu_\theta}\right)^2 \left(\frac{g_\theta}{g_\pi}\right)^2 \sim 0.16 \left(\frac{g_\theta}{g_\pi}\right)^2. \tag{7.18}$$

The ratio of the coupling constant of production $(g_\theta/g_\pi)^2$ is not known well, but from the data in the low energy regions one may regard it as the order of one. Then, the branching ratio is

$$\frac{\langle n_\theta \rangle}{\langle n_\pi \rangle} \sim \frac{1}{6}, \tag{7.19}$$

which is not so large as that of FERMI. So, it will be possible to get the correct value of the ratio only by the creation mechanism without the complicated final interaction in low energy processes.

In the original Fermi theory, owing to the picture of thermoequilibrium, the volume in which θ and π mesons are confiend must be the same. While, in our picture where the mesons are shaked off as packets the extension of the packets may be different for θ^0 and π^0. This may correspond, in the original Fermi theory, to the different equilibrium volume for θ and π.

$$\begin{cases} \Omega_\pi = \dfrac{g_\pi^2}{\mu_\pi^3} \dfrac{\mu_\pi}{\varepsilon_\pi}, \\ \Omega_\theta = \dfrac{g_\theta^2}{\mu_\theta^3} \dfrac{\mu_\theta}{\varepsilon_\theta}. \end{cases} \tag{7.20}$$

If the ratio of the extension of the packets $(g_\theta/\mu_\theta)^2/(g_\pi/\mu_\pi)^2$ is equal to 1, one obtains of course the Fermi result $(n_\theta/n_\pi) = 2$.

8. – Conclusion.

1) Fermi's and Laudau's theories of multiple production seem to be foreign elements in the field theory, but they can be formulated in the frame of the present field theory.

2) Landau's theory is at present regarded as the best, however, one can obtain equally good results by a statistical treatment of the field theory with a much simpler model.

Appendix A

As the simplest example of the spectrum function $f(k)$ which satisfies the hypotheses 1), 2) and 3) in Sect. 7, we take the following function

$$\text{(A.1)} \qquad \begin{cases} f(k_x, k_y, k_z) = \dfrac{g^2}{8\mu^3}, & \text{if} \begin{cases} -(E_0/M)\mu \leqslant k_x \leqslant E_0/M, \\ -\mu \leqslant k_y \leqslant +\mu, \\ -\mu \leqslant k_z \leqslant +\mu, \end{cases} \\ = 0 \qquad \text{otherwise.} \end{cases}$$

For simplicity we shall consider the Boltzman distribution in place of the Planck's one. Then, the multiplicity becomes

$$\text{(A.2)} \qquad \begin{cases} \langle n \rangle = \dfrac{g^2}{8\mu^3} \displaystyle\int_{-(E_0/M)\mu}^{(E_0/M)\mu} \mathrm{d}k_x \int_{-\mu}^{+\mu} \mathrm{d}k_y \int_{-\mu}^{+\mu} \mathrm{d}k_z \exp\left[-\beta(k_x^2 + k_y^2 + k_z^2 + \mu^2)\right]^{\frac{1}{2}}, \\ \sim \dfrac{g^2}{2\mu^2} \displaystyle\int_{-(E_0/M)\mu}^{(E_0/M)\mu} \mathrm{d}k_x \exp\left[-\beta\,|k_x|\right], \\ \sim g^2 \dfrac{1 - \exp\left[-(E_0/M)\beta\mu\right]}{\beta\mu}, \end{cases}$$

and

$$\text{(A.3)} \qquad E_0 = -\mu \frac{\partial\langle n\rangle}{\partial(\beta\mu)} \sim g^2\mu \left(\frac{1 - \exp\left[-(E_0/M)\beta\mu\right]}{(\beta\mu)^2} - \frac{\exp\left[-(E_0/M)\beta\mu\right]}{\beta\mu} \frac{E_0}{M} \right).$$

Putting $\beta\mu E_0/M \equiv y$, (A.3) is rewritten

$$\text{(A.4)} \qquad \frac{M^2}{g^2\mu E_0} y = \frac{1 - \exp[-y]}{y} - \exp[-y] \sim \begin{matrix} y/2 & \text{if } y \ll 1, \\ 1/y & \text{if } y \gg 1. \end{matrix}$$

But in the region $0 < y \leqslant 1$ (A.4) has no solution, because in the high energy case $M^2/g^2\mu E_0 \ll \frac{1}{2}$. So, the unique solution of (A.4) is

$$y^2 \sim \frac{g^2\mu E_0}{M^2},$$

i.e.,

$$\frac{1}{\beta\mu} = \frac{1}{g}\left(\frac{E_0}{\mu}\right)^{\frac{1}{2}}. \tag{A.5}$$

Inserting (A.5) into (A.2)

$$\langle n\rangle \sim g\left(\frac{M}{\mu}\right)^{\frac{1}{2}}\left(\frac{E_0}{M}\right)^{\frac{1}{2}}.$$

Therefore, the packet (A.1) satisfies the hypothesis 3). It is self-evident that (A.1) satisfies the hypotheses 1) and 2).

Appendix B

In order to see the relation between the Fermi theory and the statistical field theory, we shall give an example starting from the $Ps(Pv)$ interaction Hamiltonian, $H(x)$,

$$H(x) = i\,\frac{g}{\mu}\,\bar{\psi}(x)\gamma_5\gamma_\mu\psi(x)\,\frac{\partial\varphi(x)}{\partial x_\mu}\,. \tag{B.1}$$

Then, the cross section of n pion production is, as Fukuda-Takeda [6] (*),

$$\sigma_n = \frac{2\pi M}{v_0 E\mu}\left(\frac{1}{\mu^3}\,\frac{M}{E}\right)\left|\frac{\sqrt{E}(\bar{u}\gamma_5^n u_0)\sqrt{E_0}}{M}\right|^2\cdot \tag{B.2}$$

$$\cdot\,\frac{1}{n!}\prod_{i-1}^{n}\int \mathrm{d}\boldsymbol{k}_i\left(\frac{g^2}{\mu^3}\,\frac{\mu}{\varepsilon_i}\right)\delta(w-\varepsilon_1-\varepsilon_2-\ldots-\varepsilon_n)\,,$$

where $v_0 = p_0/E_0$ is the velocity of nucleons in the C.M. system, E_0 its energy. (Ep), $(\varepsilon_i \boldsymbol{k}_i)$ are energy momentum of final nucleon and of pions respectively. Now, this cross section is transformed into the representation of meson occupation numbers by enclosing the whole system in a large volume V. Denoting the phase volume $V\,\mathrm{d}\boldsymbol{k}_i/(2\pi)^3$ by Δ_i, it turns out to be

$$\sigma_n = \frac{2\pi M}{v_0 E\mu}\left(\frac{1}{\mu^3}\,\frac{M}{E_p}\right)\sum_{\substack{n_1,n_2,\ldots\\(n_1+n_2+\ldots=n)}} p(n_1, n_2\ldots)\,, \tag{B.3}$$

where

$$\left\{\begin{aligned} p(n_1,n_2\,,\ldots) &= \left|\frac{\sqrt{E}(u\gamma_5 u_0)\sqrt{E_0}}{M}\right|^2\delta(w-\varepsilon_1 n_1-\ldots)\cdot \\ &\qquad\cdot\prod_{i=1}^{n}\frac{(n_i+\Delta_i-1)!}{n_i!\,(\Delta_i-1)!}\left(\frac{(2\pi)^3}{V}\Omega_i\right)^{n_i}, \\ &\equiv \exp\left[S(n_1, n_2\ldots)\right]\delta(w-\textstyle\sum\varepsilon_1 n_i)\,, \end{aligned}\right. \tag{B.4}$$

(*) The factorization with respect to meson variables is done as an approximation.

and

$$\Omega_i = \left(\frac{g^2}{\mu^3}\right)\left(\frac{\mu}{\varepsilon_i}\right). \tag{B.5}$$

With the above preparations, the occupation numbers specifying the most probable final state are obtained by solving the variational problem

$$\sum \delta n_i \frac{\delta S(n_1, n_2 \ldots)}{\delta n_i} = 0\,, \tag{B.6}$$

with the restrictions

$$\begin{cases} \sum n_i = n\,, \\ \sum \varepsilon_i n_i = w\,. \end{cases} \tag{B.7}$$

For large occupation numbers, the problem is readily solved, resulting

$$n_i = \Omega_i \frac{\mathrm{d}\boldsymbol{k}_i}{\exp\left[\alpha + \beta\varepsilon_i\right] - 1}\,, \tag{B.8}$$

where α and β are Lagrangian multipliers determined by the conservation relations.

For extremely high energy collisions, if no restriction is imposed on the total number the « chemical potential » α may be ignored. Then, the distribution of occupation numbers in the most final state is Planck's one,

$$n(\boldsymbol{k}) = \frac{\Omega_k \mathrm{d}\boldsymbol{k}}{\exp\left[\beta\varepsilon\right] - 1}\,, \tag{B.9}$$

where the « temperature » $1/\beta$ is determined by the energy conservation

$$w = \int \frac{\Omega_k \varepsilon_k \mathrm{d}\boldsymbol{k}}{\exp\left[\beta\varepsilon\right] - 1}\,. \tag{B.10}$$

Further, Ω_k given by (B.5) may be rewritten as

$$\Omega_k = \Omega_0 \sqrt{1 - v_k^2}\,, \tag{B.11}$$

where v_k represents the velocity of the emitted pion, and

$$\Omega_0 = \frac{g^2}{\mu^3}\,, \tag{B.12}$$

is, as is easily seen, a quantity with the dimension of a volume and its mag-

nitude is comparable with Fermi's volume of thermoequilibrium

$$\Omega_0^{\mathrm{F}} \sim \frac{4}{3}\pi\frac{1}{\mu^3}\,. \tag{B.13}$$

All of these results derived from the quantized wave field by introducing statistical considerations are very similar to the results of FERMI. (B.11) is our counter part to Fermi's assumption of Lorentz contraction of the equilibrium volume

$$\Omega^{\mathrm{F}} = \Omega_0^{\mathrm{F}}\sqrt{1-v_0^2}\,. \tag{B.14}$$

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Statistical Theory of Particle Multiple Production in High Energy Nucleon Collisions.

V. S. Barašenkov, B. M. Barbašev and E. G. Bubelev

The Joint Institute for Nuclear Research
The Laboratory of Theoretical Physics - Doubna

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Contents. — **1.** Introduction. – **2.** The calculations of phase volumes. – **3.** The model of compound particles. – **4.** The results of calculations. – **5.** Discussions.

1. – Introduction.

It has been shown in a series of papers by the late professor S. Z. Belenkij [1] and his co-workers that in the energy region up to $3 \div 5$ GeV the probabilities of pion production in pion and nucleon collisions with nucleons, caclulated using Fermi's statistical theory may be brought into agreement with experiment only by taking into account strong resonance pion-nucleon interaction in the state $p = \frac{3}{2}$, $T = \frac{3}{2}$. This idea is equivalent to the introduction of a phenomenological particle N′ (« isobar ») with spin $P = \frac{3}{2}$, isotopic spin $T = \frac{3}{2}$ and mass $M_{N'} = 1.32 M_N$, where M_N is the mass of a nucleon. It is reasonable to keep the phenomenological particle N′ also in the calculations made in the higher energy region of the colliding particles. At energies $\geqslant 10$ GeV, when intensive production of strange particles is possible besides the usually

(1) S. Z. Belen'kij and A. I. Nikišev: *Žu. Éksper. Teor. Fiz.*, **28**, 744 (1955); A. I. Nikišev: *Žu. Éksper. Teor. Fiz.*, **30**, 601, 990 (1956); V. M. Maksimenko and A. I. Nikišev: *Žu. Éksper. Teor. Fiz.*, **31**, 727 (1956); S. Z. Belen'kij: *Nucl. Phys.*, **2**, 259 (1956).

used conservation of energy-momentum, isotopic spin and its projection it is necessary to take into account the conservation of strangeness and baryon number.

At present in the frame of statistical theory of multiple production one fails to take into account momentum conservation. An attempt in this direction made by FERMI (2) as it has been shown in the papers (3) is not correct. Therefore, when making the calculations we neglected momentum conservation but assumed the statistical weight of a set of particles produced in the particle collision with spins $S_1, S_2, \ldots, S_k$ to be proportional to the product $\prod_{i=1}^{k}(2S_i+1)$. Since the spins of strange particles are not exactly known at present, for the estimation of the value of the spin the minimum of all possible spins has been chosen: $S=0$ for K-mesons and $S=\frac{1}{2}$ for all the other strange particles.

In the papers (4) we have shown that one may hope to bring into agreement the results of the calculations on statistical theory with experiment only by taking into account the internal structure of the colliding nucleons, what from the mathematical standpoint is equivalent to the introduction of the additional parameter into the theory. In our theory the value of this parameter is chosen from the requirement of bringing the calculated ratio of the number of produced strange particles to the number of produced pions and nucleons into agreement with experiment.

It is of great interest to consider this questions at the maximum energies of 10 GeV achieved at present in an artificial way.

2. – The calculations of phase volumes.

In the mentioned paper the respective probabilities of pion, nucleon, antinucleon and strange particles production in the nucleon collisions with energy 7 and 10 GeV have been calculated using statistical theory. Phase volumes in the momentum space in all cases of two and three particle production have been calculated using the exact Block's formulae (5) and in the cases in which a great number of particles were produced, using the

(2) E. FERMI: *Phys. Rev.*, **81**, 683 (1951).

(3) E. L. FEJNBERG and D. S. ČERNAVSKIJ: *Dokl. Akad. Nauk SSSR*, **81**, 795 (1951); I. L. ROZENTAL' and D. S. ČERNAVSKIJ: *Usp. Fiz. Nauk*, **52**, 185 (1954).

(4) V. S. BARAŠENKOV, B. M. BARBAŠEV, E. G. BUBELEV and V. M. MAKSIMENKO: *Žu. Éksper. Teor. Fiz.* (in press); *Nucl. Phys.* (in press).

(5) M. M. BLOCK: *Phys. Rev.*, **101**, 796 (1956).

formula:

$$(1) \qquad W_{m,n}(\alpha) = 2\pi^{n+\frac{3}{2}(m-1)}\Big(\sum_{i=1}^{m} M_i\Big)^{3n+\frac{3}{2}m-4} \prod_{i=1}^{m} M_i^{\frac{3}{2}} \cdot \\ \cdot \frac{\alpha^{2n+\frac{3}{2}m-\frac{5}{2}}}{\Gamma(3n+\frac{3}{2}(m-1))} F(2n;\, \tfrac{3}{2};\, 3n+\tfrac{3}{2}(m-1);\, -\alpha)\,,$$

where m and n are respectively the number of non-relativistic and estremely relativistic particles

$$\alpha = 2\frac{E}{\sum_{i=1}^{m} M_i - 1}\,,$$

E is the energy of the colliding particles in the center of mass system;

M_i are the masses of non-relativistic particles;

F is a hypergeometrical function.

Formula (1) is the consequence of the expressions for phase volumes obtained in the papers [6,7].

For small values of the parameter α (practically $\alpha < 0.7 \div 0.8$) the hypergeometrical function F may be presented in the form of a series expansion in powers of α which is quickly convergent and convenient for the calculations:

$$(2) \qquad F(2n;\, \tfrac{3}{2};\, 3n+\tfrac{3}{2}(m-1);\, -\alpha) = \\ = \frac{\Gamma(3n+\frac{3}{2}(m-1))}{\Gamma(\frac{3}{2})} \sum_{k=0}^{\infty} C_{2n-1+k}^{k} \frac{(-1)^k \Gamma(k+\frac{3}{2})}{\Gamma(k+3n+\frac{3}{2}(m-1))} \alpha^k\,,$$

where C_{2n-1+k}^{k} are binomial coefficients.

For the numerical calculations in the case of greater values of the parameter α the function F is conveniently written in the integral form:

$$(3) \qquad F(2n;\, \tfrac{3}{2};\, 3n+\tfrac{3}{2}(m-1);\, -\alpha) = \\ = \frac{\Gamma(3n+\frac{3}{2}(m-1))}{\Gamma(n+\frac{3}{2}(m-1))\Gamma(2n)} \cdot \int_0^1 \frac{t^{2n}(1-t)^{n+\frac{3}{2}m-\frac{5}{2}}}{(1+\alpha t)^{\frac{3}{2}}}\,\mathrm{d}t\,.$$

At great number of the produced particles (> 3) the formulas (1)–(3) approximate well enough the exact expression of phase momentum volume and in cases when the part of the produced particles is deliberately « intermediate »,

(6) I. L. Rozental' and V. M. Maksimenko: *Žu. Èksper. Teor. Fiz.* (in press).

(7) S. Z. Belen'kij, I. L. Rozental', A. I. Nikišev and V. M. Maksimenko: *Usp. Fiz. Nauk* (in press).

i.e. cannot be referred to the category of extremely relativistic and non-relativistic particles. In the cases when the precision of the formulae (1)–(3) appeared to be unsufficient the interpolation methods were applied.

As comparative calculations have shown, the arising error is insignificant (*).

3. – The model of compound particles.

As well as in the calculations made earlier ([4]) sharp difference with experiment ([8,9]) arises if we assume all the particles to be produced in the same space volume V_π or V_K (respectively with radius equal to the Compton wavelength of the pion or the K-meson). The choice of other values of the volume V leads to analogous results. The contradiction with experiment may be removed if we assume the different kinds of elementary particles to be produced in different space volumes. It can be understood in view of the formation of indivisible « compound particles » by the colliding particles where all the energy of the particles is concentrated in the interaction energy, but not in the proper energy of the particles ([10]). The dimensions of the regions of interaction energy localization are specific for every concrete type of interaction.

We have made the calculations for two different assumptions in respect to the character of elementary particle interaction:

1) If we assume that the hyperons interact with pions as strongly as the nucleons, then these particles must be produced in the same volume V_π. Corresponding antiparticles as well as isobars and antiisobars will be produced in the same volume. For K^- and K^+-meson we assume a considerably less strong interaction with the pion field, so that these particles will be produced in the volume V_K. The interaction energy of K^- and K^+-mesons with the baryon field which is produced by the pion field in the volume V_π may be neglected in

(*) It is worth noting that the use for the calculations of $W_{m,n}$ of closed expressions obtained in the paper ([7]) is very difficult because of catastrophic disappearance of terms when subtracting in these expressions terms of nearly equal magnitude. This circumstance is clearly expressed also when calculating the numerical coefficients, determining the expressions $W_{m,n}$ in ([7]).

([8]) S. Goldhaber: *Proceedings of the Fifth Annual Rochester Conference* (1955); L. van Rossum and L. T. Kerth: *Bull. Amer. Phys. Soc.*, **1**, 385 (1956); W. W. Chupp *et al.*: *Suppl. Nuovo Cimento*, **4**, 359 (1956); S. J. Lindebaum, and C. L. Luke Yuan: *Phys. Rev.*, **105**, 1931 (1957); G. Harris, J. Orear and S. Taylor: *Phys. Rev.*, **101**, 1214 (1956).

([9]) G. D. Rochester and C. C. Butler: *Rep. Progr. Phys.*, **16**, 364 (1953).

([10]) D. I. Blohincev: *Usp. Fiz. Nauk*, **61**, 137 (1957).

comparison with the interaction energy with the baryon field, localized in the volume V_K (i.e., in the « core » of the compound « particle »).

The structure of the compound particle thus created corresponds to the hypothesis of « global interaction » of baryons with the pion field [11] suggested recently by SCHWINGER and GELL-MANN. It is closely connected with the nucleon model proposed by Academician I. E. TAMM [12]. In Tables II-IV this case of the choice of space volumes is marked by the subscript (1).

2) If we assume that strange particles in contrast to the nucleons interact with the pion field comparatively weakly and the interaction energy of these particles with nucleons produced in the volume V_π by the pion field is negligibly small in comparison with the baryon field localized in the « core » of the compound particle V_K, then all the strange particles will be produced in the volume V_K; while the pions, nucleons, isobars and the corresponding antiparticles will be produced in the volume V_π.

Such a choice of the compound-particle structure corresponds to the hypothesis of nucleon structure suggested by NEGANOV [14] or PEASLEE [15] and does not contradict the nucleon model suggested by I. E. TAMM.

In Tables II-IV this case of choosing the compound-particle structure is marked by the subscript (2).

Because of strong interaction in the regions V_π and V_K statistical equilibrium can be established. The statistical theory of multiple production suggests that the time of the collision is greater than the time of the establishing of such an equilibrium.

In the collision of two high velocity nucleons the strong interaction region V_K may be considered as forming as a result of the collision of nucleon cores and the region V_π as formed by the collision of external regions of nucleons and the extension of the region V_K.

In more strict theory together with the possible introduction of « the strong interaction volumes » V_K and V_π in the expressions for the statistical weights the interaction constants must be explicitly taken into account.

The difference in the interaction constants for different kinds of particles in the considered variants of theory is shaped to some extent by the introduction of different volumes.

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[11] V. JU. VOTRUBA, M. JA. DANYSZ and F. E. LAW: *Information on VII Annual Rochester Conference* (Doubna, May 1957).

[12] I. E. TAMM: *Report at Theoretical Physics Congress held in Seattle* (USA, September 1956); *Žu. Éksper. Teor. Fiz.*, **32**, 178 (1957); V. M. KOPROV and V. S. BARAŠENKOV: *Žu Éksper. Teor. Fiz.* (in press); E. G. BUBELEV: *Žu Éksper. Teor. Fiz.* (in press).

[13] M. A. MARKOV: *Dokl. Akad. Nauk SSSR*, **101**, 64 (1953).

[14] B. S. NEGANOV: *Žu. Éksper. Teor. Fiz.* (in press).

[15] D. C. PEASLEE: *Phys. Rev.*, **105**, 1934 (1957).

4. – The results of the calculations.

When making the calculations we used the values of masses of elementary particles, which are given in Table I in terms of the nucleon masses:

TABLE I.

	π	$K; \bar{K}$	$N\bar{N}$	$\Lambda; \bar{\Lambda}$	$\Sigma; \bar{\Xi}$	$N'\bar{N}'$	$\Xi\bar{\Xi}$
M_i	0.15	0.526	1	1.19	1.27	1.32	1.41

The results of the calculations of respective probability of possible reactions in proton-proton and proton-neutron collisions are given in Tables II and III. In these tables W_i is the probability of the reaction in percentages,

$$\overline{W}_i = \sum_{n=0}^{\max} W_i(n) \,, \tag{4}$$

the total probability of the reaction:

$$\left(\begin{array}{c} \mathrm{pp} \\ \mathrm{or} \\ \mathrm{pn} \end{array} \right) \to (A \ldots B \cdot n\pi) \qquad n = 0; 1 \ldots n_{\max}. \tag{5}$$

Since the complete analysis of all the particles produced in the particle collision by the charge signs is rather difficult, in Table III the division by the charge signs is fulfilled only for strange particles. For the calculations of the statistical weights in the isotopic space the tables of the paper ([16]) were used; the values of the charge factor necessary for the division by the charge signs were taken from the papers ([1]). In the cases when the probability of the reaction is very small the division by the charge signs was made only for antihyperons since these reactions are of great interest in connection with the study of the possibilities of discovery and experimental study of antihyperons with the big accelerator at the Joint Institute for Nuclear Research, at Doubna.

In Table IV the average values of the number of pions, nucleons, antinucleons, K and $\bar{K}$-particles, hyperons Y and antihyperons $\bar{Y}$ produced in (pp) and (pn) collisions at different energies of the incident nucleon E and at two above discussed asusmptions on the compound-particle structure are

([16]) V. S. BARAŠENKOV and B. M. BARBAŠEV: *Nuovo Cimento* (submitted to publ.).

TABLE II.

	(pp); 7 GeV		(pp); 10 GeV		(pn); 7 GeV		(pn); 10 GeV	
	W_1 %	W_2 %	W_1 %	W_2 %	W_1 %	W_2 %	W_1 %	W_2 %
n	$NNn\pi$							
0	0.084	0.097	0.0190	0.022	0.14	0.163	0.032	0.037
1	3.36	3.88	1.13	1.26	3.43	3.99	1.19	1.35
2	17.8	20.5	8.92	10.2	19.3	22.4	9.74	11.2
3	36.0	41.4	26.2	30.0	34.4	39.7	25.5	29.3
4	22.2	25.6	29.1	33.3	22.0	25.5	29.0	33.4
5	6.05	6.98	14.9	17.1	5.78	6.70	14.4	16.6
6	0.49	0.56	2.82	3.23	0.47	0.54	2.71	3.12
7			0.58	0.65			0.43	0.50
	$N\Lambda Kn\pi$							
0	0.68	0.035	0.28	0.0145	1.13	0.059	0.47	0.025
1	2.34	0.120	1.33	0.068	2.00	0.103	1.18	0.061
2	1.77	0.090	1.79	0.091	1.90	0.098	1.95	0.099
3	0.23	0.012	0.67	0.033	0.24	0.012	0.72	0.036
4	0.022	0.0^211	0.07	0.0^24	0.022	0.0^211	0.07	0.0^24
	$N\Sigma Kn\pi$							
0	1.23	0.064	0.52	0.027	1.33	0.069	0.57	0.029
1	4.04	0.207	2.44	0.124	4.36	0.225	2.66	0.136
2	3.22	0.165	3.63	0.183	3.16	0.162	3.64	0.185
3	0.34	0.017	1.28	0.064	0.33	0.017	1.25	0.063
4	0.031	0.0^215	0.14	0.0^27	0.030	0.0^215	0.14	0.0^27
	$2N\bar{K}Kn\pi$							
0	0.022	0.025	0.020	0.023	0.028	0.032	0.026	0.030
1	0.069	0.079	0.074	0.084	0.063	0.073	0.069	0.080
2	0.037	0.043	0.087	0.099	0.038	0.045	0.091	0.104
3	0.0^234	0.0^239	0.032	0.077	0.0^233	0.0^238	0.034	0.076
	$2\Lambda 2Kn\pi$							
0	0.0^220	0.0^524	0.0^224	0.0^529	0.0^233	0.0^541	0.0^241	0.0^550
1	0.0^336	0.0^643	0.0^212	0.0^515	0.0^338	0.0^647	0.0^213	0.0^516
	$\Lambda\Sigma 2Kn\pi$							
0	0.0^240	0.0^549	0.0^281	0.0^548	0.0^243	0.0^553	0.0^288	0.0^553
1	0.0^386	0.0^510	0.0^238	0.0^545	0.0^393	0.0^511	0.0^241	0.0^550

TABLE II (*continued*).

	(pp); 7 GeV		(pp); 10 GeV		(pn); 7 GeV		(pn); 10 GeV	
	W_1 %	W_2 %	W_1 %	W_2 %	W_1 %	W_2 %	W_1 %	W_2 %
n	$2\Sigma 2Kn\pi$							
0	$0.0^{2}36$	$0.0^{5}44$	$0.0^{2}73$	$0.0^{5}88$	$0.0^{2}39$	$0.0^{5}48$	$0.0^{2}79$	$0.0^{5}96$
1	$0.0^{3}55$	$0.0^{6}7$	$0.0^{2}33$	$0.0^{5}39$	$0.0^{3}55$	$0.0^{6}7$	$0.0^{2}33$	$0.0^{5}40$
	$N\Xi 2Kn\pi$							
0	$0.0^{2}66$	$0.0^{3}26$	0.012	$0.0^{3}49$	$0.0^{2}84$	$0.0^{3}34$	0.016	$0.0^{3}63$
1	$0.0^{2}65$	$0.0^{3}20$	0.021	$0.0^{3}82$	$0.0^{2}61$	$0.0^{3}17$	0.020	$0.0^{3}78$
2	$0.0^{3}41$	$0.0^{5}4$	$0.0^{2}64$	$0.0^{3}19$	$0.0^{3}41$	$0.0^{5}5$	$0.0^{2}65$	$0.0^{3}19$
	$3N\bar{N}n\pi$							
0	0.072	0.083	0.52	0.059	0.092	0.106	0.66	0.76
1	$0.0^{2}18$	$0.0^{2}21$	1.34	1.53	$0.0^{2}19$	$0.0^{2}26$	1.23	1.42
2			1.03	1.17			1.09	1.26
3			0.05	0.05			0.05	0.05

given. The data for the energies $E = 3$ and $E = 5$ GeV have been taken from the Tables of the papers [3].

In Tables II-IV the exponent of the zero means the number of zeros after the comma, for instance: $0.0^{3}57 = 0.00057$.

5. – Discussion.

As it was mentioned above the taking into account of the compound particle structure is quite necessary for bringing into agreement theory with experiment [8,9]. However, in spite of the fact that the probabilities of the strange particle production by both variants of the choice fo the compound particle structure are different by one order it is difficult to determine which of the considered variants is more close to the experiment. An interesting consequence from the assumption on the compound particle structure is the maximum of the strange particle ratio to the number of pions and nucleons, in the energy region of 7 GeV.

The cause of it consists in the probability of the reaction $(N\bar{N}) \to (N\Lambda Kn\pi)$ and $(N\bar{N}) \to (N\Sigma Kn\pi)$ decreases when passing from $E = 5 \div 7$ GeV to $E = 10$ GeV, the probability of the reaction with greater number of K-mesons, for instance

TABLE III.

Reaction	(pp) 10 GeV		(pn) 10 GeV	
	W_1 %	W_2 %	W_1 %	W_2 %
$2N\Lambda\bar{\Lambda}$	0.11	$0.0^{2}15$	0.19	$0.0^{2}26$
$2N\Lambda\bar{\Lambda}\pi$	0.07	$0.0^{2}10$	0.05	$0.0^{3}6$
$2N\Lambda\bar{\Sigma}(\bar{\Lambda}\Sigma)$	0.15	$0.0^{2}21$	0.17	$0.0^{2}23$
$2N\Lambda\bar{\Sigma}(\bar{\Lambda}\Sigma)\pi$	0.06	$0.0^{3}8$	0.07	$0.0^{3}9$
$2N\bar{\Sigma}\Sigma$	0.19	$0.0^{2}26$	0.21	$0.0^{2}29$
$2N\bar{\Sigma}\Sigma\pi$	0.05	$0.0^{3}6$	0.05	$0.0^{3}6$
$2N\bar{\Xi}\Xi$	$0.0^{2}93$	$0.0^{3}13$	0.012	$0.0^{3}16$
$N\Lambda\bar{\Xi}\Sigma$	$0.0^{2}52$	$0.0^{5}24$	$0.0^{2}56$	$0.0^{5}27$
$N\bar{\Xi}2\Lambda$	0.016	$0.0^{5}74$	0.027	$0.0^{4}13$
$3N\bar{\Lambda}\bar{K}$	$0.0^{3}27$	$0.0^{4}14$	$0.0^{3}35$	$0.0^{4}18$
$2N\bar{N}\Lambda K$	$0.0^{3}81$	$0.0^{4}41$	$0.0^{2}104$	$0.0^{4}53$
$3N\bar{\Sigma}\bar{K}$	$0.0^{3}13$	$0.0^{5}67$	$0.0^{4}74$	$0.0^{5}73$
$2N\bar{N}\Sigma K$	$0.0^{3}40$	$0.0^{4}20$	$0.0^{3}43$	$0.0^{4}22$

Reaction	pp		(np)	
	$\bar{W}_1$ %	$\bar{W}_2$ %	$\bar{W}_1$ %	$\bar{W}_2$ %
$N\Lambda K^{+}n\pi$	3.05	0.156	2.65	0.137
$N\Lambda K^{0}n\pi$	1.99	0.102	2.65	0.137
$N\Sigma^{+}K^{+}n\pi$	2.14	0.110	1.17	0.060 3
$\Sigma^{+}K^{0}$	2.14	0.110	1.86	0.096 0
$\Sigma^{0}K^{+}$	1.79	0.092	1.56	0.080 6
$\Sigma^{0}K^{0}$	1.10	0.056	1.56	0.080 6
$\Sigma^{-}K^{+}$	1.35	0.069	1.86	0.096 0
$\Sigma^{-}K^{0}$	0.33	0.017	1.17	0.060 3
$2N\bar{K}^{0}K^{+}n\pi$	0.040	0.046	0.030	0.035
$\bar{K}^{0}K^{0}$	0.034	0.040	0.035	0.040
$K^{-}K^{+}$	0.034	0.040	0.035	0.040
$K^{-}K^{0}$	0.018	0.021	0.040	0.035
$N\Xi^{0}K^{+}K^{+}n\pi$	$0.0^{2}21$	$0.0^{4}84$	$0.0^{3}83$	$0.0^{4}33$
$\Xi^{0}K^{+}K^{0}$	$0.0^{2}43$	$0.0^{3}17$	$0.0^{2}37$	$0.0^{3}15$
$\Xi^{-}K^{+}K^{+}$	$0.0^{2}21$	$0.0^{4}84$	$0.0^{2}18$	$0.0^{4}73$
$\Xi^{0}K^{0}K^{0}$	$0.0^{2}10$	$0.0^{4}38$	$0.0^{2}18$	$0.0^{4}73$
$\Xi^{-}K^{+}K^{0}$	$0.0^{2}19$	$0.0^{4}76$	$0.0^{2}37$	$0.0^{3}15$
$\Xi^{-}K^{0}K^{0}$	0	0	$0.0^{3}83$	$0.0^{4}33$

TABLE III (*continued*).

Reaction	(pp)		(np)	
	$\overline{W}_1$ %	$\overline{W}_2$ %	$\overline{W}_1$ %	$\overline{W}_2$ %
$n\pi \cdot N\Lambda K^+$	2.48	0.126	2.20	0.112
K^0	1.66	0.084	2.20	0.112
$n\pi \cdot N\Sigma^+ K^+$	1.86	0.094	1.08	0.091
$\Sigma^+ K^0$	1.86	0.094	1.64	0.128
$\Sigma^0 K^+$	1.58	0.080	1.40	0.111
$\Sigma^0 K^0$	1.04	0.052	1.40	0.111
$\Sigma^- K^+$	1.23	0.063	1.64	0.128
$\Sigma^- K^0$	0.44	0.022	1.08	0.091
$n\pi \cdot 2N\overline{K}^0 K^+$	0.065	0.080	0.047	0.059
$\overline{K}^0 K^0$	0.053	0.066	0.057	0.071
$K^- K^+$	0.053	0.067	0.057	0.071
$K^- K^0$	0.030	0.039	0.047	0.059
$n\pi \cdot N\Xi^0 K^+ K^+$	$0.0^2 67$	$0.0^3 25$	$0.0^2 36$	$0.0^3 14$
$\Xi^0 K^+ K^0$	0.013 4	$0.0^3 50$	0.011 7	$0.0^3 44$
$\Xi K^+ K^+$	$0.0^2 67$	$0.0^3 25$	$0.0^2 58$	$0.0^3 23$
$\Xi^0 K^0 K^0$	$0.0^2 42$	$0.0^3 16$	$0.0^2 58$	$0.0^3 23$
$\Xi^- K^+ K^0$	$0.0^2 83$	$0.0^3 32$	0.011 7	$0.0^3 44$
$\Xi^- K^0 K^0$	$0.0^3 4$	$0.0^4 1$	$0.0^2 36$	$0.0^3 14$

Reaction	(pp)		(np)	
	$\overline{W}_1$ %	$\overline{W}_2$ %	$\overline{W}_1$ %	$\overline{W}_2$ %
$n\pi \cdot 2N\Lambda\overline{\Sigma}^-(\overline{\Lambda}\Sigma^+)$	0.136	$0.0^2 184$	0.073	$0.0^3 98$
$\overline{\Sigma}^0(\overline{\Lambda}\Sigma^0)$	0.062	$0.0^3 82$	0.087	$0.0^2 118$
$\overline{\Sigma}^+(\overline{\Lambda}\Sigma^-)$	0.016	$0.0^3 21$	0.073	$0.0^3 98$
$n\pi \cdot 2N\Sigma\overline{\Sigma}^-$	0.122	$0.0^2 164$	0.084	$0.0^2 113$
$\overline{\Sigma}^0$	0.075	$0.0^2 101$	0.087	$0.0^2 117$
$\overline{\Sigma}^+$	0.042	$0.0^3 56$	0.085	$0.0^2 114$
$n\pi \cdot (2\Sigma + 2\Lambda)\overline{\Xi}^-$	0.019	$0.0^5 89$	0.016	$0.0^5 76$
$\overline{\Xi}^0$	$0.0^2 20$	$0.0^6 91$	0.016	$0.0^5 76$
$n\pi \cdot 2N\Xi\overline{\Xi}^-$	$0.0^2 70$	$0.0^4 95$	$0.0^2 60$	$0.0^4 82$
$\overline{\Xi}^0$	$0.0^2 23$	$0.0^4 32$	$0.0^2 60$	$0.0^4 82$

TABLE IV.

E	π		$\bar{N}$		K		$\bar{K}$		Y		$\bar{Y}$	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
	(pp)-collision											
3	1.92	2.00	—	—	0.056	0.0^226	0.0^592	0.0^596	0.056	0.0^226	—	—
5	2.45	2.67	—	—	0.13	0.0^269	0.0^341	0.0^347	0.13	0.0^264	—	—
7	2.88	3.11	0.0^374	0.0^385	0.14	0.0^286	0.0^213	0.0^215	0.14	0.0^271	—	—
10	3.35	3.60	0.029	0.033	0.12	0.0^290	0.0^221	0.0^228	0.13	0.0^263	0.0^287	0.0^312
	(pn)-collision											
3	1.88	1.99	—	—	0.065	0.0^231	0.0^411	0.0^412	0.065	0.0^231	—	—
5	2.40	2.63	—	—	0.14	0.0^274	0.0^346	0.0^353	0.14	0.0^269	—	—
7	2.84	3.08	0.0^394	0.0^211	0.15	0.0^290	0.0^213	0.0^215	0.15	0.0^275	—	—
10	3.30	3.56	0.030	0.035	0.13	0.0^294	0.0^222	0.0^229	0.14	0.0^266	0.010	0.0^313

though it increases with energy cannot, however, considerably change the total probability, because K-mesons are produced in the volume V_K and the probability of their production is strongly suppressed in comparison with pions and nucleons.

The decrease of strange particles number in the high energy region ($E > 10$ GeV) would be in contradiction with the cosmic ray experiments [9].

However, at high energies, in variant 1 the reactions of the type:

$$(NN) \to 2N2\Lambda n\pi) \; ; \qquad (NN) \to (2N2\Sigma n\pi) \, , \quad \text{etc.}$$

can give considerable contribution and the number of strange particles may increase. More detailed consideration in the case of variant 2 is also necessary. These questions are being studied now.

The number of the produced antinucleons depends slightly on the variant chosen for the compound-particle structure and is approximately equal to one antinucleon per two-three tousand pions for the energy $E = 7$ GeV and one antinucleon per one-two hundred nucleons at the energy $E = 10$ GeV. This result can be put in agreement with experimental data by the order of magnitude: one antinucleon per $6 \cdot 10^4 \div 5 \cdot 10^5$ of the negative pions obtained when bombarding copper targets with protons energy $E = 6.2$ GeV [17,18],

[17] O. CHAMBERLAIN, E. SEGRÈ, C. WIGAUD and T. YPSILANTIS: *Phys. Rev.*, **100**, 947 (1955); H. H. HECKMAN, F. M. SMITH and W. H. BARKAS: *Bull. Amer. Phys. Soc.*, **1**, 186 (1956).

[18] W. H. BARKAS *et al.*: *Phys. Rev.*, **105**, 1037 (1957).

if we take into account anormally small antinucleon range in nuclear matter in respect to the annihilation ([18]),

$$l_{\text{an}} = \frac{1}{\sigma_{\text{an}} n} \sim \frac{\hbar}{m_\pi c} .$$

Due to this fact antinucleons produced on one of the nucleons in the nucleus of Cu practically annihilate on the following nucleon, and from the nucleus only antinucleons produced on the nucleus surface may emerge.

Taking into account that the incident nucleon interacts in the Cu nucleus with the « tube » containing on the average $3 \div 4$ nucleons and that $\sigma_{\text{el}}/\sigma_{\text{inel}} \sim 0.25$ we obtain the probability of the incident nucleon reaching the last nucleon in the tube with small energy loss $W \sim 5 \cdot 10^{-2}$.

The basic contribution to the production of antinucleons on the nucleus of Cu is given by these nucleons, because the cross-section of the antinucleon production quickly decreases with decreasing energy (see Tables II-IV). Due to the above considerations the ratio of the number of the produced antinucleons to the number of pions decreases up to the magnitudes expermentally observed ([19,20]).

* * *

In conclusion we express our gratitude to D.I. BLOHINCEV for his interest and discussions. We are grateful to L. G. ZASTAVENKO for fruitful discussions and V. M. MAKSIMENKO and I. L. ROZENTAL' who presented us the results obtained in the papers ([6,7]) before their publication.

([19]) It is worth taking into account that the nucleon in the nucleus of Cu possesses a Fermi energy of 25 MeV; due to this the colliding nucleons in the experiments ([17,18]) possess the same energy in the center of mass system as for a free nucleon collision at an energy $E = 7$ GeV.

([20]) An analogous phenomenon is well-known in the atomic reactor theory. It is called « block-effect ». In our case the nuclei of Cu play the part of « blocks ».

NOTE TECNICHE

Il laboratorio per raggi cosmici del «Sabbione».

E. Fiorini, C. Giori e C. Succi

Istituto di Scienze Fisiche dell'Università - Milano
Istituto Nazionale di Fisica Nucleare - Sezione di Milano

C. Moriggia

Società Edison-Volta - Milano

(ricevuto il 16 Agosto 1957)

Contenuto. – 1. Introduzione. – 2. La capanna. – 3. Sistemazione generale del laboratorio. – 4. Comunicazioni e servizi. – 5. Il locale della camera. – 6. Attuale impiego del laboratorio.

1. – Introduzione.

Presso l'Istituto di Scienze Fisiche della Università di Milano si conducono, da più di un decennio, ricerche sulla radiazione cosmica. Per lo studio delle interazioni nucleari a grande energia (> 10 GeV) è stata in questi ultimi anni progettata e costruita una grande camera di Wilson avente dimensioni di ingombro di $156 \times 156 \times 98$ cm^3, destinata, almeno nei prossimi anni, a ricerche a quota di montagna [1].

Poichè un conveniente montaggio di tutta l'apparecchiatura avrebbe richiesto un locale di non meno di 50 m^2 di superficie, non è stato possibile usufruire dell'esistente Laboratorio della Testa Grigia [2], essendo quest'ultimo già in precedenza prenotato da alcuni gruppi di ricercatori di altre Università.

Si è allora cercato, in altre località di montagna, un edificio avente requisiti tali da poter venire adattato a laboratorio. La Società Edison, da noi interpellata nella persona dell'Ing. Candiani, ci segnalò e successivamente ci fece visitare le seguenti località: la centrale idroelettrica dell'Alpe Devero (Novara), il cantiere di costruzione degli impianti idroelettrici sull'alto Chiese (Trento) ed il cantiere della diga del Sabbione (Novara).

La quota favorevole di quest'ultima località, l'esistenza in loco di servizi

(1) E. Fiorini, R. Giacconi e C. Succi: *Nuovo Cimento* (in corso di stampa) (1957).
(2) G. Bernardini, G. Longo e E. Pancini: *Ric. Scient.*, **18**, 91 (1948).

per il trasporto di materiali pesanti (fino a 50 quintali), la possibilità di accesso garantita tutto l'anno, l'esistenza di un collegamento telefonico diretto con Milano ed altre condizioni favorevoli di cui si dirà più avanti, hanno fatto scegliere la diga del Sabbione come sede del nostro Laboratorio.

Per gentile interessamento del Dott. Bobbio, Consigliere Delegato della Società Edison, l'Ing. Justi, Direttore Tecnico della Società Edison-Volta, autorizzava l'uso e l'impiego, a tempo indefinito, di una capanna del cantiere posta a quota 2518 m s.m. a circa una cinquantina di metri dalla stazione terminale della funivia Morasco-Sabbione.

2. – La capanna.

La capanna adibita a Laboratorio, che aveva ospitato dirigenti e tecnici preposti alla costruzione della diga, ha pianta rettangolare con lati di 15.20 m e 5.20 m; l'altezza è di metri 2.50. Le coordinate geografiche sono all'incirca 46° 25′ 30″ di latitudine Nord e 4° 08′ 52″ di longitudine Ovest (Monte Mario). L'asse maggiore della capanna giace nei quadranti Nord-Ovest e Sud-Est, e forma un angolo di circa 40° con la direzione Nord-Sud.

Posta su di un piccolo rilievo roccioso sovrastante il lago e la diga del Sabbione e, prospiciente il grande ghiacciaio dell'Hochsandhorn la capanna (Fig. 1) è costruita su di un basamento in muratura di pietrame con malta di cemento, di altezza variabile da circa 50 cm a circa 3 m, ed ha un muro longitudinale a sostegno del pavimento.

Fig. 1. – La capanna Laboratorio e la stazione della funivia.

Le pareti perimetrali sono formate da pannelli di 100 × 250 cm², aventi uno spessore di circa 10 cm, con una intercapedine di 6 cm riempita di vermiculite. La parete esterna dei pannelli, collegati fra loro a mezzo di montanti a doppio T, è di legno di larice, quella interna di abete.

Il pavimento, sopraelevato rispetto alla roccia sottostante, è sostenuto a mezzo di travi e realizzato con pannelli da 100 × 300 cm², disposti longitudinalmente.

La copertura, a falde inclinate di 30°, sostenuta con capriate di tavole di legno poste ad un metro di interasse, è realizzata con tavole di 2.5 cm di spessore, ricoperte con cartone catramato e lamiera ondulata da 6/10 di mm. Per aumentare la stabilità contro i venti che, durante l'inverno, soffiano impetuosi da nord-est e sud-ovest, la capanna è puntellata e controventata con otto tiranti in corda di acciaio.

3. – Sistemazione generale del laboratorio.

Per permettere alla capanna di ospitare la grande camera di Wilson, si son resi necessari alcuni lavori di adattamento e variazioni alla disposizione preesistente dei locali.

Data la imperfetta sutura dei pannelli, si è provveduto a migliorare l'isolamento termico della capanna, per ridurre gli effetti delle variazioni esterne di temperatura, rivestendo le pareti interne con carta e con un foglio di politene da 0.1 mm; sopra il politene è stato poi disposto un rivestimento di faesite. Il pavimento è stato ricoperto con faesite e su questa è stato applicato del linoleum.

Per sostenere alcune parti pesanti dell'apparecchiatura ed isolare la capanna dalle vibrazioni del grosso compressore della camera, si son dovuti costruire sulla roccia alcuni basamenti di calcestruzzo. Si son pure ancorate direttamente alla roccia, a mezzo di robuste putrelle di ferro, le due macchine fotografiche stereoscopiche della camera di Wilson, per impedire che le vibrazioni dovute alle espansioni della camera danneggiassero la bontà dei fotogrammi.

La disposizione dei locali e la loro funzione sono mostrate in Fig. 2.

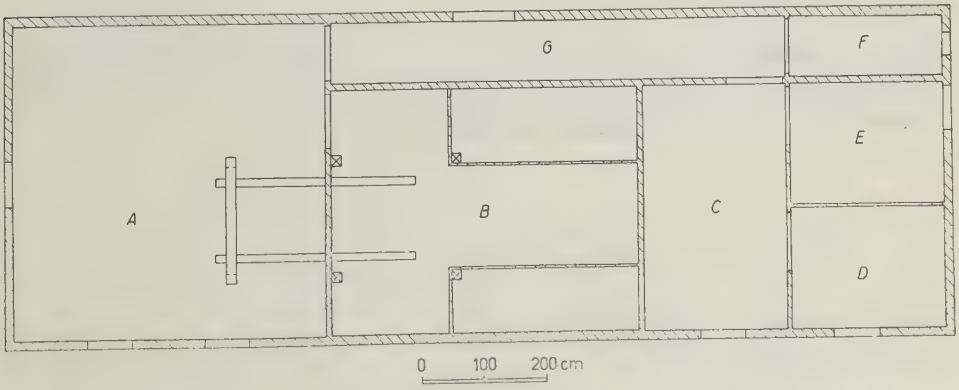

Fig. 2. - Pianta del laboratorio: *A*) Laboratorio; *B*) Locale per le camere; *C*) Soggiorno; *D*) Camera da letto; *E*) Camera da letto; *F*) Bagno; *G*) Corridoio d'ingresso.

Il locale *A* è il Laboratorio propriamente detto e contiene, oltre all'apparecchiatura della camera, i quadri di distribuzione dell'energia elettrica e l'impianto di termostatazione. Parte dello stesso locale è adibita ad officina meccanica.

Nel locale *B*, destinato a contenere le camere, le lampade per l'illuminazione ed i condensatori sono disposti quattro travi di larice, di sezione quadrata con 25 cm di lato, che sostengono, ad un'altezza variabile, un piano di putrelle di ferro, di 170 × 180 cm² di superficie, capace di portare fino a 100 quintali di materiale.

In corrispondenza di tale piano di sostegno, per avere maggiore disponibilità di spazio, il soffitto è stato sopraelevato di circa 50 cm.

Il locale *C* è adibito a soggiorno (Fig. 3) ed i locali *D* ed *E* sono adibiti a camere da letto: la stanza *D* contiene un letto, la *E* due cuccette.

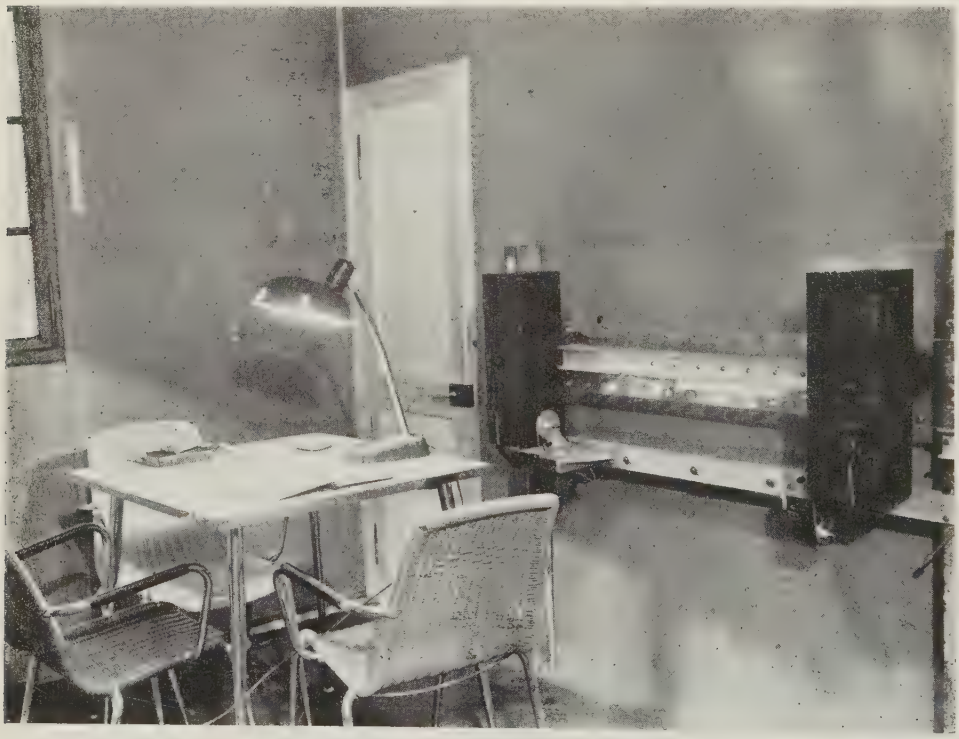

Fig. 3. – Scorcio del locale adibito a soggiorno.

F è il bagno, dotato di lavabo con acqua corrente calda e fredda, doccia e W.C.

4. – Comunicazioni e servizi.

Al laboratorio si può accedere per mezzo della funivia Morasco-Sabbione: il percorso viene compiuto in circa 24 minuti primi; la base della funivia è col-

legata al cantiere di Valdo, e di qui a Domodossola (Novara) da una strada carrozzabile.

D'inverno, quando il tratto Valdo-Morasco si chiude per la neve, la stazione inferiore della funivia può venire raggiunta attraverso le gallerie in cui sono sistemate le condotte forzate: il tempo che normalmente si impiega a superare il tragitto è di circa un'ora e mezzo.

I materiali pesanti vengono trasportati al Laboratorio mediante una teleferica che, partendo dalla località Morasco, raggiunge il Sabbione in prossimità della stazione della funivia. Durante la stagione invernale materiali anche pesanti, ma non troppo ingombranti, possono venire trasportati alla teleferica attraverso le gallerie.

Il laboratorio è dotato di energia elettrica che viene fornita da una cabina, distante dalla capanna una cinquantina di metri, per mezzo di un trasformatore trifase da 7 500÷8 000 volt primari e 230÷240 volt secondari. La potenza attualmente disponibile, di circa 250 kVA, è aumentabile a 325 kVA impiegando un secondo trasformatore di riserva, collegabile in parallelo.

Nel Laboratorio un trasformatore universale trifase da 20 kVA ed un trasformatore universale monofase da 15 kVA permettono di alimentare due altre reti interne di distribuzione a 160 volt. Una quarta rete interna alimentata da un raddrizzatore al selenio da 600 watt distribuisce corrente continua a 24 volt.

Una tubazione di 2 pollici di diametro, in gran parte sotterranea, fornisce acqua al Laboratorio prelevandola da una sorgente alimentata dal ghiacciaio del Blindenhorn, posta a circa 1 km dalla capanna.

Il riscaldamento della capanna è ottenuto elettricamente a mezzo di 12 stufe da 500 watt, alimentabili a coppie in serie o in parallelo e pilotate da termointerruttori bimetallici in modo da mantenere nell'interno una temperatura intorno ai 20 °C. Per fronteggiare interruzioni prolungate nella distribuzione di energia elettrica la capanna è pure dotata di riscaldamento ed illuminazione a gas liquido.

5. – Il locale della camera.

Perchè le caratteristiche del locale che ospita la camera siano tali da garantire un buon funzionamento è necessario stabilizzarne la temperatura entro qualche decimo di grado centigrado, anche quando le condizioni esterne sono particolarmente sfavorevoli.

Per ottenere ciò, oltre a migliorare l'isolamento termico del pavimento e della parte esposta al vento, si sono impiegati tre termoconvettori aventi ciascuno una superficie utile di 20 m^2 disposti come è indicato in Fig. 1 ed alimentati con acqua circolante in ciclo chiuso a 20 °C.

L'acqua proviene da un serbatoio di 450 dm^3 di capacità, in cui un termometro tipo Vertex, sensibile al decimo di grado pilota due riscaldatori blindati aventi una potenza regolabile da 500 a 1 000 watt.

Una pompa della portata di 6 litri al minuto primo rimescola l'acqua nel serbatoio. Una seconda pompa della portata di 20 litri al minuto primo, pilotata da un secondo Vertex di eguale sensibilità, posto nel locale della camera, fa circolare l'acqua nei termoconvettori. Con il dispositivo impiegato la temperatura è risultata stabile entro 3/10 di grado centigrado.

Per ostacolare la formazione di gradienti di temperatura nel locale della camera si sono disposti tre altri ventilatori. Ad un metro e mezzo di altezza dal pavimento ed in vicinanza della camera il gradiente orizzontale di temperatura è risultato inferiore a 0.2 °C/m ed il gradiente verticale inferiore a 0.5 °C/m.

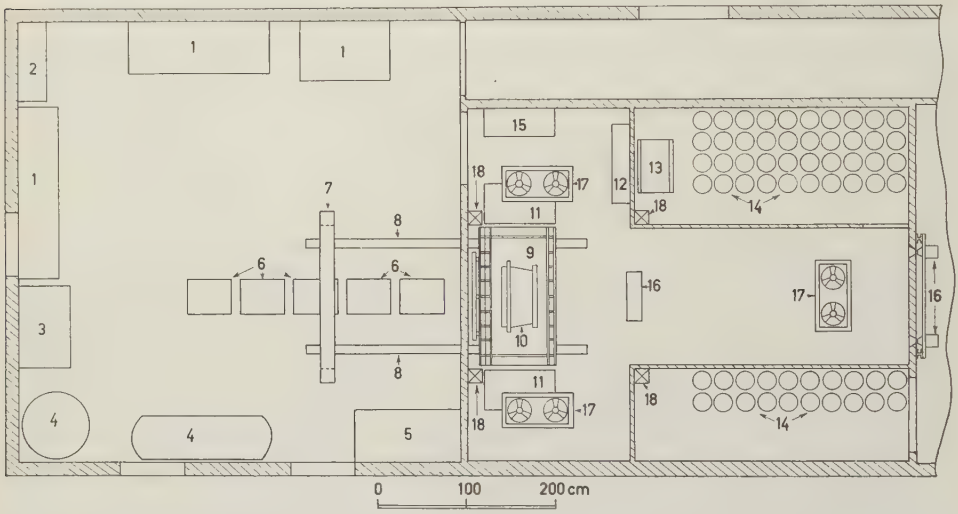

Fig. 4. – Disposizione degli apparecchi nel laboratorio: 1) Attrezzatura d'officina; 2) Centrale elettrica; 3) Compressore; 4) Serbatoi per aria; 5) Termostato per acqua; 6) Apparecchi elettronici; 7) Ponte di sollevamento; 8) Rotaie; 9) Camera di Wilson grande; 10) Camera di Wilson piccola; 11) Lampade a flash; 12) Dispositivo di sicurezza pei flash; 13) Alimentatore elettronico; 14) Batteria di condensatori; 15) Lampada a incandescenza; 16) Macchine fotografiche; 17) Termo-convettori; 18) Supporti per piano di sostegno.

Per ridurre eventuali riflessioni di luce, dannose per la fotografia, le pareti del locale contenente la camera sono state ricoperte con un drappo nero.

Per facilitare le operazioni di montaggio e smontaggio, la camera è stata posta su di un carrello della portata di 10 quintali, scorrevole su rotaie fissate su una serie di otto pilastrini di calcestruzzo. Con semplice trazione la camera viene trasportata, attraverso un'apposita porta quadrata, nel locale adibito a laboratorio sotto un ponte di sollevamento costituito da un paranco montato su travi di legno.

6. – Attuale impiego del laboratorio.

Attualmente al laboratorio si avvicendano alcuni ricercatori e tecnici delle Università di Milano e di Pisa [3], costituiti in un gruppo per una ricerca il

[3] M. Conversi, E. Fiorini, C. Giori, C. Rubbia, C. Succi e R. Torelli: *Progetto di una esperienza per la ricerca di particelle di massa* 550 m_e; *Conferenza internazionale sulle particelle elementari*, Padova, 1957.

cui scopo è di accertare l'esistenza nella radiazione cosmica delle particelle di massa 550 m_e, segnalate recentemente (*) da ALIHANIAN *et. al.*

L'apparecchiatura è costituita da un telescopio di due grandi camere di Wilson (Fig. 5), associate in un selezionatore elettronico di particelle, ed ha

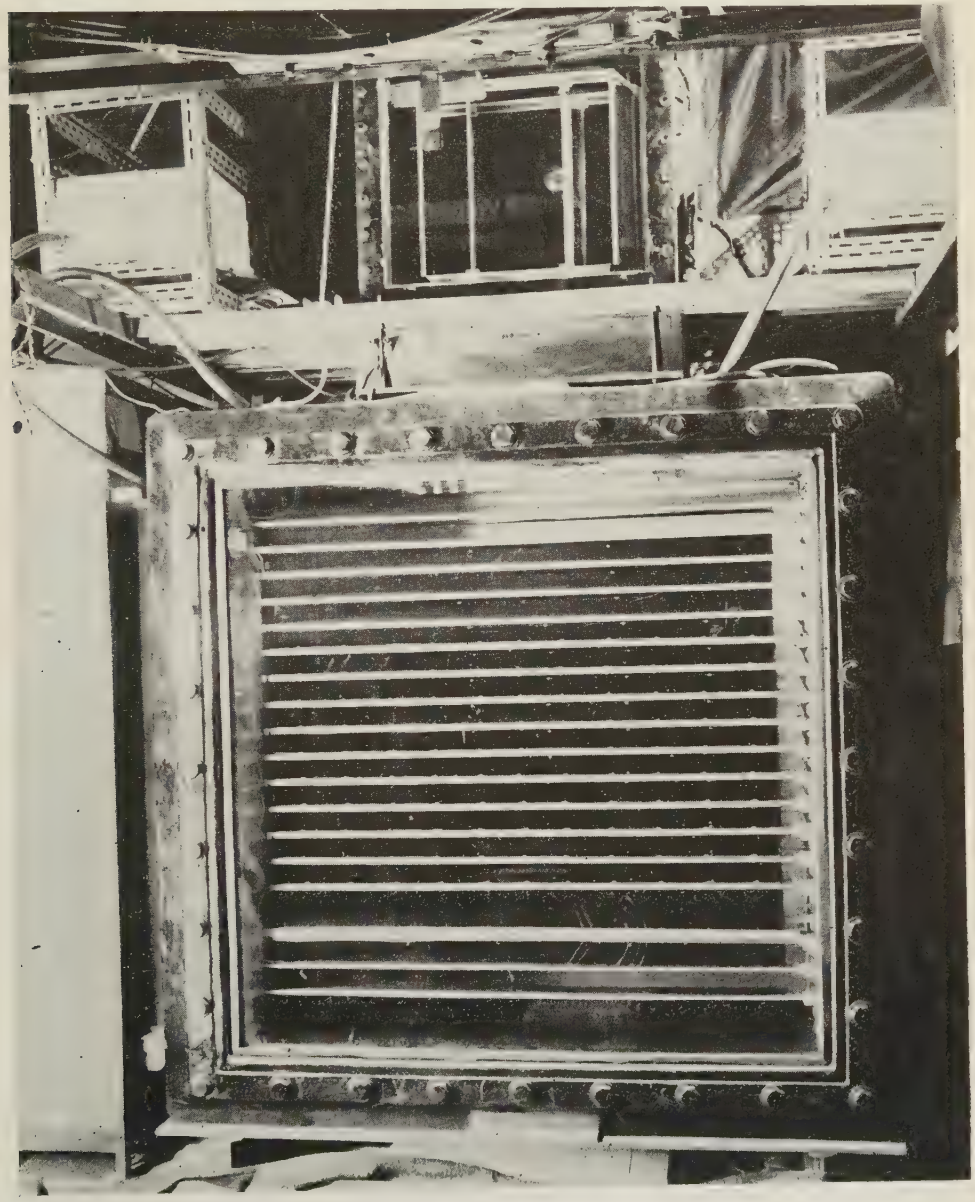

Fig. 5. – Il telescopio di camere di Wilson.

(*) Comunicazione privata del prof. M. CONVERSI.

lo scopo di permettere la determinazione della massa, del decadimento e della vita di queste particelle.

Le Fig. 6 e 7 mostrano alcuni aspetti del laboratorio durante la fase di montaggio degli apparecchi.

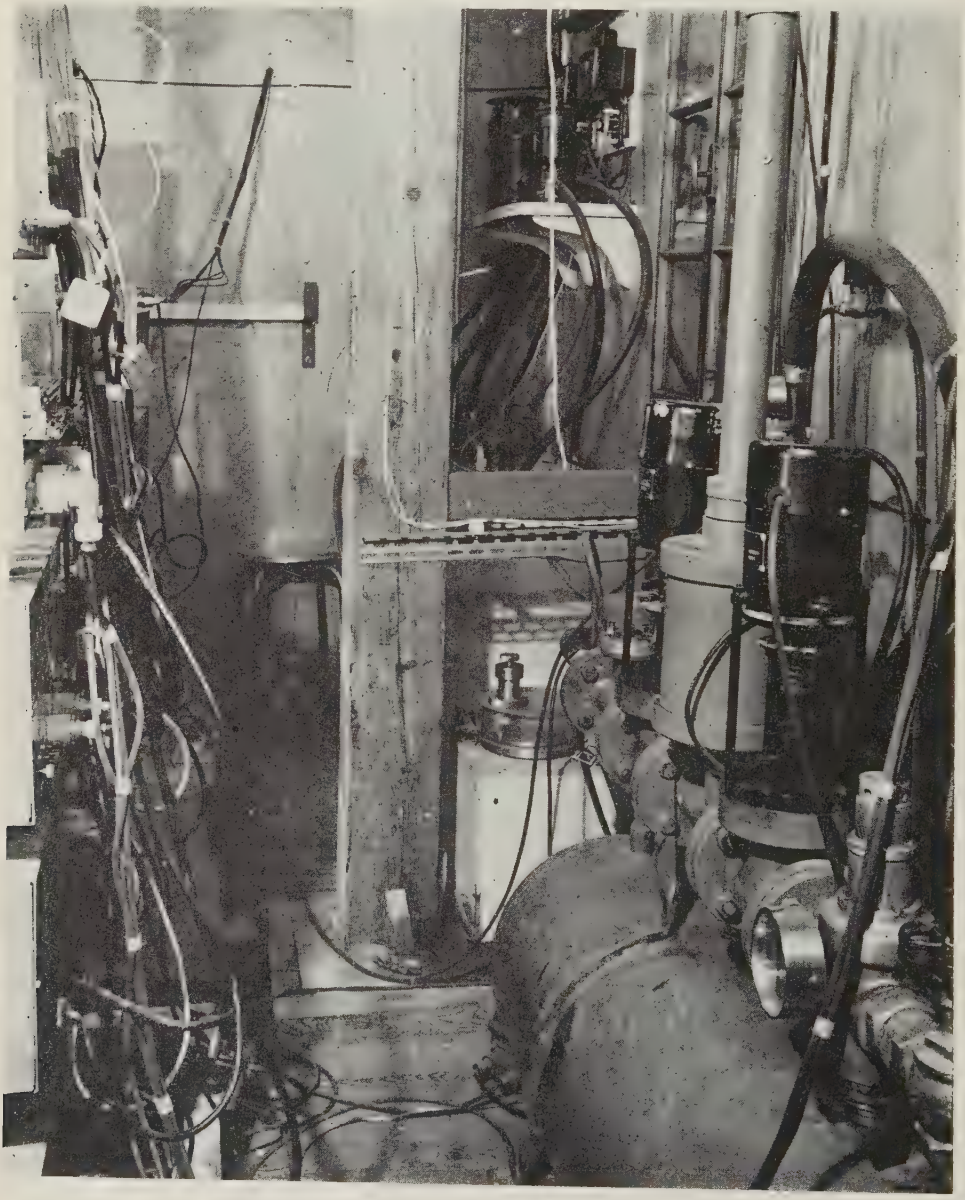

Fig. 6.

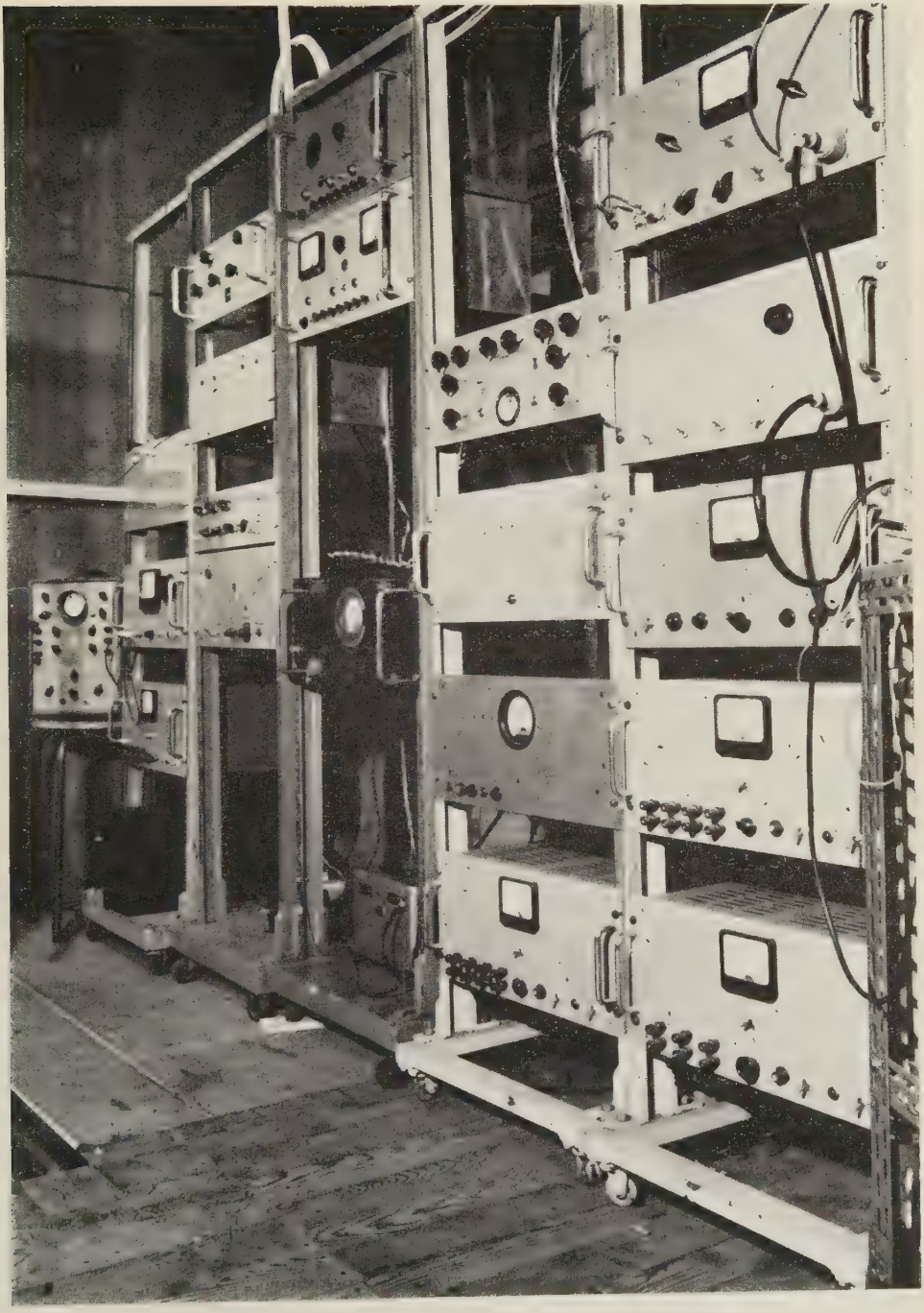

Fig. 7.

* * *

La costruzione del Laboratorio è stata possibile grazie al generoso e continuo appoggio ricevuto dal Gruppo degli Amici dell'Istituto di Scienze Fisiche, in particolare dalle Società Edison ed Edison-Volta, A.N.I.D.E.L., Pirelli, Linoleum, Agipgas alle quali vogliamo esprimere la nostra più viva riconoscenza.

È nostro gradito dovere ringraziare particolarmente il Dott. C. Bobbio, Consigliere Delegato della Società Edison, per l'interessamento e la sollecitudine con cui ha seguito il nostro lavoro; il Dott. Ing. L. Justi, Direttore Tecnico della Società Edison-Volta, per le sue numerose concessioni; ed il Dott. Ing. M. Calciati, Direttore di Zona della Società Edison-Volta, per il suo cordiale e generoso aiuto.

Al Prof. G. Polvani, Direttore dell'Istituto di Scienze Fisiche dell'Università di Milano, ed al Prof. P. Caldirola, Direttore della Sezione di Milano dell'I.N.F.N., esprimiamo la nostra più viva riconoscenza per il continuo appoggio ed incoraggiamento datoci durante l'allestimento del Laboratorio.

SUPPLEMENTO AL VOLUME VII, SERIE X
DEL NUOVO CIMENTO

N. 1, 1958
1° Trimestre

A Microwave Electron Spin Resonance Spectrometer (*).

M. B. PALMA-VITTORELLI and M. U. PALMA
Istituto di Fisica dell'Università - Palermo

(ricevuto il 19 Agosto 1957)

CONTENTS. — **1. Introduction.** – **2. The magnet.** – **3. The magnetic field.** 1. Design considerations. 2. The power stage. 3. The error-signal channel. 4. Composite loop behaviour. 5. Modulation and measurement of the magnetic field. – **4. The microwave system.** 1. Design considerations. 2. The apparatus. – **5. Detection and recording of the magnetic absorption signal.**

1. – Introduction.

The importance of the electron spin resonance technique has been growing as higher sensitivities and precisions have been reached. The equipment here described reaches a sensitivity which is considerably higher than the one of similar commonly described equipments. The high stabilization of both the magnetic field and the microwave frequency yields a precision which is essentially limited by the width of the absorption lines (in the case which is of interest for us, i.e. solid state). The result is achieved by the use of relatively simple circuitry. The use of a rotating magnet enables ready and accurate measurements of anysotropy and saturation to be performed.

As at is well known, one is interested not only in the position of the absorption maxima, but also in the whole plots of the real and imaginary components (χ' and χ'') of the complex magnetic susceptibility. Far from saturation, the complex susceptibility is an analytical function of the frequency,

(*) This work forms part of a program of research undertaken with the financial support of the Consiglio Nazionale delle Ricerche, through the Comitato Nazionale per la Fisica, for which the A.A. wish to express their sincere gratitude. The work has been performed at the suggestion of Prof. M. SANTANGELO, whose large and valuable advice, criticism and help are gratefully aknowledged by the AA.

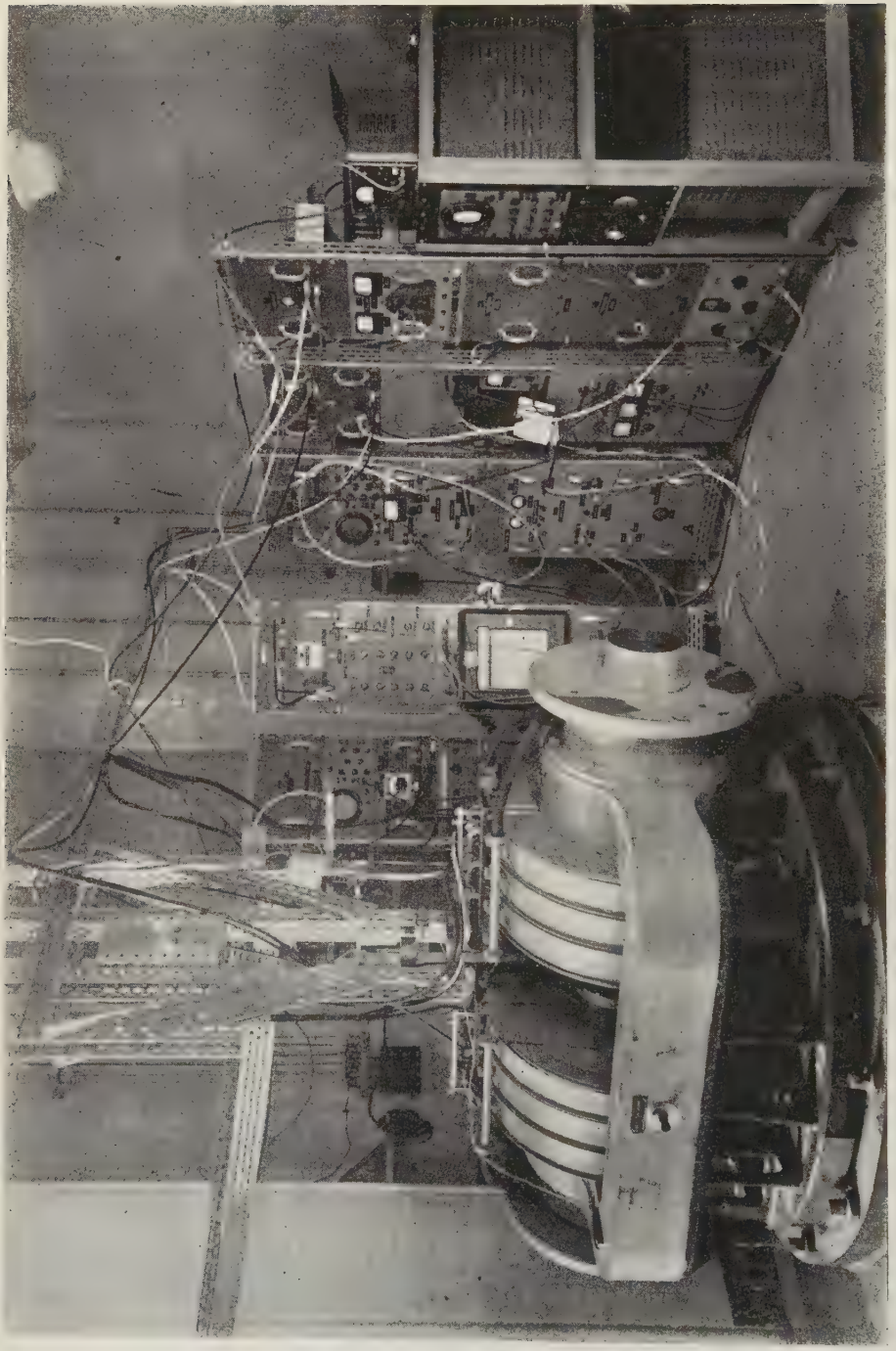

Fig. 1 Photograph of the apparatus.

and χ' and χ'' are related by the Kramers-Krönig relations, which may be found by means of a complex contour integration (*).

The two plots (vs. the quasi-static magnetic field and vs. the microwave frequency) are not equivalent. In fact the simple relation which gives the maximum of the absorption curve, i.e. $h\nu = g\beta H$ (where h = Planck's constant, β = Bohr magneton, g = const = spectroscopic splitting factor) does not always hold, in the sense that, if for instance one operates at constant frequency, the ratio ν/H_{max} may depend on the frequency at which one operates (for instance, when a pre-splitting is present). Moreover, the line shape and width may be noticeably affected by the spin-lattice relaxation time, which is in turn H-dependent. It is a usual practice, in order to avoid remarkable experimental difficulties to get plots of χ' and χ'' vs. H, although plots vs. ν would give better information on the different quantities involved (relaxation times, exchange interaction etc.) and on their magnetic field- and temperature-dependence. In our equipment too, the measurements are performed at constant frequency, but more than one frequency is available in order to compare experimental data; further, the available power in the K-band (about 600 mW) renders frequency multiplication feasible.

The electron spin resonance spectrometer which will be described in the following has the following characteristics: magnetic field homogeneity better than $3 \cdot 10^{-4}\%$ in the sample volume (solid state samples) and of the order of $10^{-2}\%$ in the volume of the whole cavity; magnetic field stability of the order of $2 \cdot 10^{-3}\%$ short term, and of the order of $5 \cdot 10^{-3}\%$ long term (and a suitable precision in the measurement, using proton resonance); magnetic field sweep linearity, of the order of 0.1%; frequency stability of the order of $10^{-4}\%$ ultimate sensitivity, at room temperature, with a signal-to-noise ratio of about 4:1, (using optimum modulation amplitude) about $10^{13} \cdot \Delta H$ magnetic centers (where ΔH is the line-width in Gauss).

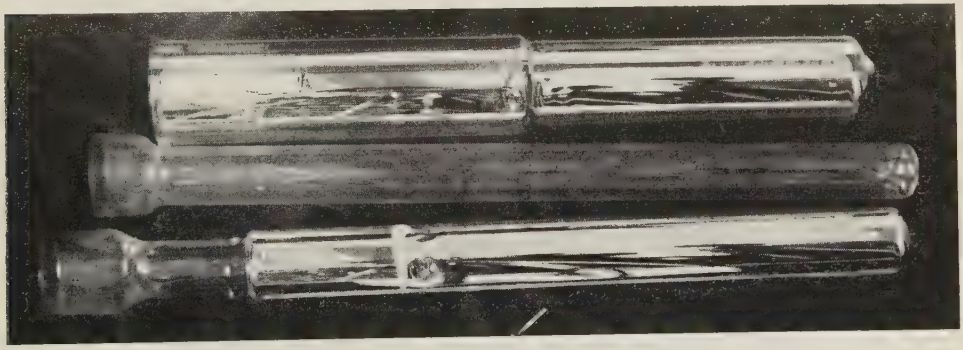

Fig. 2. – Photograph of the cryostats for low temperature measurements.

(*) The presence of poles in the area encircled by the integration contour invalidates the relationships and corresponds to saturation in magnetic absorption (see, for instance, ref. [1]). Much in the same way, in circuit theory, the Bode relationship is found, which is of fundamental importance in feedback loop design and which relates gain characteristics and phase-shifts in four-terminal networks. In this case, the presence of poles corresponds to the presence of non-minimun-phase-networks.

The above performances refer to a typical case of 3 cm wavelength and room temperature. The sensitivity may be further increased by an estimated factor of several hundreds, using the available system for low temperatures (down to liquid helium) and the K-band set up.

The equipment is partially shown in Fig. 1; the low temperature system is shown in Fig. 2.

2. – The magnet.

The electromagnet was designed to fulfil at the same time the requirements of electron spin resonance experiments and of a good general purpose laboratory magnet (*).

It has a horizontal double yoke, rotating around a vertical axis (i.e. around an axis orthogonal to the direction of the magnetic field in the gap) and the rotation angle is easily measurable. The poles are of a diameter of 25 cm. The gap is continuously variable within large limits, and a mechanical system allows the symmetry around the center to be accurately preserved. There are a number of cylindrical precision machined pole pieces, with ground faces, some having Rose shims ([2]) suitable for air gaps up to about 11 cm and others having independent shims ([3]). In every position corresponding to useful air gaps, the poles are in good mechanical balance so that the coincidence of their axes within strict tolerances is secured. Anyway, an alignement is required, for each setting, using three spacers and possibly tiny shims: the control is made mechanically first (using a micrometer) and then using the proton probe. In order to decrease the residual inhomogeneity, a saturation of the magnet in the reverse direction, followed by a slow cycling of the field ([4]) has been found useful. In this way the values specified in Sect. 1 have been obtained.

In the coil system design, two requirements have been satisfied: high efficiency defined as the ratio (induction field in the air gap $\times$ gap)/ampere-turns; and low input power, in order to allow the use of the Laboratory storage battery, as a power supply, and to simplify the problems concerning control, stabilization and cooling. The first feature has been obtained using the double yoke and a very good iron quality; this renders the required number of ampere-turns reasonably small. In addition, the required power has been kept low by the use of a large copper mass, in accord with the fact that this power is proportional to $r^2(Ni)^2/M$ (where M is the copper mass of the coils,

(*) The magnet has been constructed by the Ansaldo San Giorgio S.p.A. using Terni iron of the type T11S: we are here very pleased to thank the Terni S.p.A., for the generous gift of the iron to this laboratory; the Ansaldo San Giorgio S.p.A. for accurate construction, and Dr. Ing. A. Lotti, of the Ansaldo San Giorgio, for very valuable advice on practical details and the design of the cooling system.

([1]) A. M. Portis: *Phys. Rev.*, **91**, 1071 (1953).

([2]) M. E. Rose: *Phys. Rev.*, **53**, 715 (1938).

([3]) J. D. Bjorken and F. Bitter: *Rev. Sci. Inst.*, **27**, 1005 (1956).

([4]) K. D. Bowers, R. A. Kamper and R. B. D. Knight: *Journ. Sci. Inst.*, **34**, 49 (1957).

r the average radius of the turns and Ni the ampere-turns). Accordingly, the coil system consists of six independent sections, each of 100 turns, 5 ohms, three on each pole. The sections are separated from the yoke and each other by hollow pancakes in which the cooling water circulates. Moreover, suitable discharges are provided, in order to protect the magnet against harmful voltage surges.

Roughly, up to 10 kilogauss, using a gap of about 10 cm, the required current is about 1 ampere per kilogauss, when the six sections are series-connected.

The possibility to rotate the magnet, together with the use of the cylindrical resonant cavities which will be described in Sect. 4·2, allows precise and ready measurements of anisotropy. The sample is located at the center of the cavity, on a holder which may rotate around a diameter of the circular section. The radio-frequency magnetic field H_{rf} on the sample is directed along the cylindrical axis of the cavity. The geometry of the system is shown in Fig. 3*a*. The directions of H, of the rotation axis of the sample, and of the rotation axis of the magnet, are rendered orthogonal to each other by the use of a Gauss self-collimating system which will be described in the following.

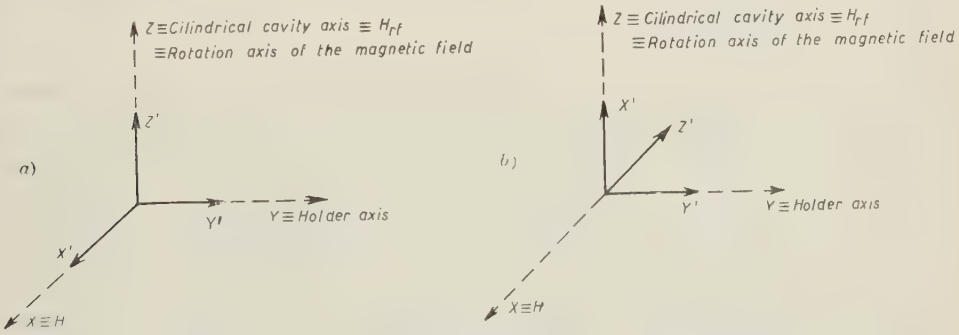

Fig. 3. – Illustration of the method for anysotropy measurements (see text).

Measurements in the $x'y'$-plane of the crystal may be made simply by rotating the magnet around the z-axis (see Fig. 3*a*). Measurements in the $z'x'$-plane may be made by rotating only the crystal around the y' axis and keeping H along the x direction. Finally, measurements in the $y'z'$-plane may be made by rotating first the sample holder by $\pi/2$ around the y-axis (thus obtaining the orientation shown in Fig. 3*b*), and then by rotating the magnet. The three sets of measurements are sufficient for the determination of the anisotropy in magnetic resonance absorption, often independently of the initial knowledge of the directions of the principal magnetic axes ([5]).

In order to orientate the cavity together with the sample holder, as shown in Fig. 3*a*, a Gauss self-collimating telescope is used, which fits into a small hole. This hole has been precision-machined in the yoke, in a direction which

([5]) M. B. Palma-Vittorelli, M. U. Palma, D. Palumbo and M. Santangelo: *Nuovo Cimento*, **2**, 811 (1955).

passes through the center of the gap and is orthogonal to both the pole and the rotation axes. The sample holder has a head, having a flat face orthogonal to the rotation axis. On this face a small mirror is secured, which is used to obtain the self-collimation condition, thus rendering the holder axis coincident with the y-axis. A diametral line is engraved on the mirror: if a suitable microscope is used in the hole instead of the self-collimation telescope, the rotation of the holder may be measured (*).

This method eliminates the lack of accuracy in anisotropy measurements due to the setting and re-setting of the sample in three known orthogonal directions. Furthermore, it allows in any case the measurements to be made, in one single run, with a consequent saving of time and refrigerants, (if used).

One extra feature of the rotating magnet is the possibility of ready and precise measurements of saturation parameters, operating at a constant microwave level ([6]). Indeed, using a cavity in which H_{rf} (on the sample) lies in the xy-plane of Fig. 3*a*, one may measure the departure from the law which gives the absorbed power (proportional to $H_{rf}^2 \sin^2 \widehat{H,H_{2j}}$), as the magnet is being tated. The saturation parameter may be then easily computed.

3. – The magnetic field.

3·1. *Design considerations.* – In order to stabilize and control the magnetic field of an electromagnet which is energized by a motor-generator set, the motor-generator itself may be used as the power stage of the control loop, by feeding the amplified error-signal into the generator field-control winding. Such a system, which was originally developed by Sommers, Weiss and Halpern ([7]), has been widely applied ([8-10,1]), in spite of considerable difficulties arising from the non-linearity of the generator and large phase-shifts.

If a fixed voltage power supply or a large storage battery is available, as in our case, a very simple stabilization system may be obtained by the use of a variable impedance in series with the magnet. Such a variable impedance, controlled by a servo-loop, may correct the possible fluctuations or shifts of the magnetic field. For instance, one may use as a variable impedance element a number of tubes in parallel, the grids of which are driven by the amplified error-signal. This is the principle of operation largely used in many low-power stabilized power supplies of common use.

If the dominant lag of the loop, up to the frequency corresponding to a unitary open-loop gain, is given by the magnet, the loop itself is inherently stable,

(*) The whole optical system has been constructed by the Officine Galileo S.p.a.; the help of Prof. G. Giotti is gratefully ocknowledged.

([6]) G. Feher: *Bell. Syst. Tech. Journ.*, **26**, 449 (1957).

([7]) H. S. Sommers, P. R. Weiss and W. Halpern: *Rev. Sci. Inst.*, **20**, 244 (1949); **22**, 612 (1951).

([8]) J. R. Wolff and M. S. Freedman: *Rev. Sci. Inst.*, **22**, 736 (1951).

([9]) M. W. P. Strandberg, M. Tinkham, I. H. Solt and C. F. Davis: *Rev. Sci. Inst.*, **27**, 596 (1956).

([10]) F. T. Hedgcock and F. Hunt: *Rev. Sci. Inst.*, **27**, 970 (1956).

([11]) H. W. Bode: *Network Analysis and Feedback Amplifiers Design* (New York, 1945).

([12]) J. G. Thomason: *Linear Feedback Analysis* (London, 1955).

since the slope of its controlled high-frequency cut-off characteristic atmost a value of — 6 db/octave ([11,12]). In such a case the phase-margin would be 90°. It follows that we may use at most in the error signal channel, either a single lag having a time constant comparable with the one of the magnet, or a number of lags of smaller time constants up to the reduction of the phase-margin to, say, 10° ([11,12]).

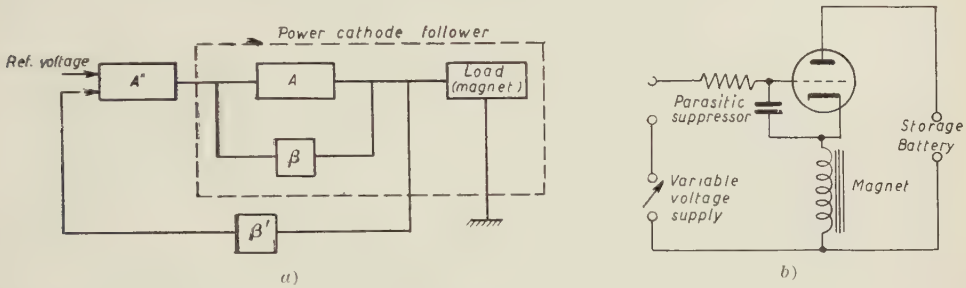

Fig. 4. – Block-diagram of the composite loop for stabilization and control of the magnetic field, and detail of the cathode-follower power stage (for tubes and parasitic suppressors, see text).

In order to further narrow the bandwidth of the error-signal channel (i.e. to narrow the bandwidth and to obtain a related reduction of drifts and noise of the high-gain amplifier) and in order to get the extra advantages which will be specified later, we have used a composite loop. The block-diagram of the composite loop is shown in Fig. 4*a*. The part of the circuit within the dotted line (power stage) is shown in detail in Fig. 4*b*, where a single tube stays for a bank of up to 96 parallel EL34 tubes. The gain of this stage, supposing that the magnet is mounted as a plate load, is indicated by A. The cathode-follower action is symbolized by the introduction of the feedback amount β. The load consists of the magnet. The error-signal channel is actuated by: the feedback β', the error-signal amplifier, which has a gain A', and a matching cathode-follower.

Let us think of the composite loop in terms of a single loop, the feedback amount of which we will label by β_{equ}. Let A_{equ} be its gain without feedback, and let

$$G = \frac{A_{\text{equ}}}{1 - A_{\text{equ}}\beta_{\text{equ}}}, \tag{1}$$

be the equivalent closed-loop overall gain. By equivalent open-loop gain we will mean the product $A_{\text{equ}}\beta_{\text{equ}}$.

One may easily see, by inspection of Fig. 4*a*, that

$$G = \frac{AA'}{1 - AA'(\beta' + (\beta/A'))}, \tag{2}$$

so that

$$A_{\text{equ}}\beta_{\text{equ}} = AA'(\beta' + \beta/A'). \tag{3}$$

The above equation will be written in its explicit form in Sect. 3·4. There we will see that, for a given high-frequency cut-off characteristic of the equivalent open-loop gain, we may let A' drop faster, with the result that the bandwith of the error-signal amplifier may be narrowed or the gain may be increased.

Let us now look for the behaviour of the different stages.

3·2. *The power stage.* – The power stage, as shown in Fig. 4*b*, contains up to 96 EL34 tubes in parallel, having the magnet as a load, in a cathode-follower mounting. The power supply is the laboratory storage battery which gives up to 250 V. The heating current is supplied from a stable d.c. source. A very well stabilized, variable voltage supply is used in order to bring the tubes in the proper working range, and to give the possibility of a manual control of the field. Furthermore, it allows an easy switching of the magnet, without harmful voltage surges.

The cathode-follower mounting offers the following extra advantages: since its voltage gain is about unity, larger noises and drifts are allowed, at constant overall gain, at the input (i.e. at the output of the high gain error-signal amplifier). Further, it makes the phase-shifts in the power stage negligible. It is to be noted anyway that the cathode-follower action reduces the voltage-to-voltage phase-shifts, while the magnetic field-to-voltage phase-shifts are determined only by the time constant of the magnet.

An elementary analysis shows that in the entire frequency range in which we are interested (parasitic suppressors disregarded), it is:

$$A = \mu \frac{R(R+\varrho) + \omega^2 L^2 + jL\omega l}{(R+\varrho)^2 + \omega^2 L^2} , \tag{4}$$

where μ is the amplification factor and ϱ the plate resistance of the EL34's which are regarded as a single tube, and R, L are respectively the resistance and self-inductance of the magnet. Eq. (4) shows that the gain A is a function of the number of tubes used, of the resistance of the magnet (depending on the connection of the different sections), and of the air gap in the magnet. For typical values, related to 50 triode-mounted tubes, series-connection of the magnet sections, and a gap of less than 10 cm, $\varrho = 12\ \Omega$, $\mu = 6$, $R = 30\ \Omega$, $L = 100$ H; A is almost frequency-independent and may be taken as equal to about 5, up to $\omega \approx 1000$. At higher values of ω the parasitic suppressor's time constant becomes of importance and determines the high frequency cut-off characteristic, which therefore has a slope of -6 db/octave at most.

Using the mounting exemplified in Fig. 4*b*, we have:

$$\beta = -1 , \tag{5}$$

but lower absolute values are obtained if some of the sections are mounted as plate load.

3·3. *The error-signal channel.* – Two different types of error-signal channels have been developed, using respectively a magnetic field and a current discriminator.

In the first, the principle developed by KIP and ARNOLD ([12 bis]) and then applied also by STRANDBERG and co-workers ([9]) is used: a signal is given by a coil which rotates in the magnetic field at a speed of 25 r. p. s. This signal is compared in a Poggendorf bridge (potentiometer), with the one given by a reference generator driven by the same axis; the latter signal may be conveniently adjusted in amplitude and phase, for proper operation of the difference-voltage detector. The latter is a cathode-follower, in order to preserve the linearity of the system, and gives the difference between the flip-coil voltage and the fraction of the reference generator voltage corresponding to a given setting of the potentiometer. The latter is a helical potentiometer, which may be set manually or swept by linking its shaft to the drive-shaft extension of the Speedomax recorder (see Sect. **5**). A simple switching system allows an easy change of the sweep range and amplitude, by putting suitable resistances in series and in parallel with the potentiometer. For a given potentiometer setting, a field fluctuation or shift modulates the 25 Hz carrier generated in the rotating coil and gives rise to an error-signal which corrects the field. Inversely, a given change of the potentiometer setting (i.e. of the used fraction of the reference voltage), gives rise, through the loop, to the wanted shift in the field value. The error-signal is amplified in a low frequency feedback amplifier and phase-detected in a Brown Converter. The RC filter which follows the phase-detector may have a time constant up to the order of the magnet's and its bandwith is in any case small compared with the bandwidth of the amplifier. So doing, the RC filter is the dominant lag in the error-signal channel. (Should the side-bands undergo appreciable phase-shifts in the a.c. amplifier, they would be opposite in sign, and the phase-detector action could not cancel them, but would give in this case the average of their absolute values ([13])). This system offers the advantage of giving a good linearity when sweeping the field from very low values up to a number of kilogauss. For higher field values, the linearity is no longer preserved ([9]). Moreover, in order to have negligible phase-shifts in the amplifier, one has either to have a large bandwidth, or to limit the gain. This, together with the fact that the brushes of the flipping coil are noisy, gives in any range a limit of the order of half a gauss in the field stabilization even with the use of the composite loop. The flipping coil provides in any case a system for an approximate measurement of the field ([12 bis]).

When a higher stability is wanted, a d.c. error-signal channel is used. A high-rating 1 Ω resistor in series with the magnet gives a voltage proportional to the current, and having the same phase of it. The same helical potentiometer bridge as above is now used to compare this voltage with a fraction of a reference battery voltage. The resulting error-signal is then amplified in a d.c. amplifier. The high-frequency cut-off characteristic of the channel is again determined by some form of RC filter like the one following the phase-detector in the first case. The d.c. amplifier bandwidth again has to be larger than the filter's bandwidth, but now, compared with the flipping coil case, a given high-frequency cut-off slope, in db/octave, and a given amplifier gain correspond to a much smaller bandwidth.

([12 bis]) R. D. ARNOLD and A. F. KIP: *Phys. Rev.*, **75**, 1199 (1949).

([13]) M. B. PALMA-VITTORELLI, M. U. PALMA and D. PALUMBO: *Nuovo Cimento*, **6**, 1211 (1957).

In both cases, the resulting error-signal is fed into the power stage through a matching cathode-follower stage, which uses two push-pull tubes in order to have a larger output voltage swing.

The expressions of β' are the same in both cases, and the numerical values too, as long as magnetic saturation is negligible:

$$\beta' = \beta'_0 \frac{1 - j\omega\tau_M}{1 + \omega^2\tau_M^2}, \tag{6}$$

(where $\beta'_0 \approx -1/30$ is the β' value corresponding to $\omega = 0$ and τ_M is the magnet time constant which, for the typical values given above, is about 3 s). Eq. (6) may be easily obtained, since β' is equal to the ratio between the non-amplified error-signal voltage and the magnet voltage variation to be corrected.

The A' expressions and values are likewise the same for both cases, namely,

$$A' = A'_0 \frac{1 - j\omega\tau_F}{1 + \omega^2\tau_F^2}, \tag{7}$$

where A'_0 is the d.c. (or carrier-frequency) gain, and τ_F the time constant of the dominant lag.

5·4. *Composite loop behaviour.* – Using eqs. (4) to (7), one gets

$$A_{\text{equ}}\beta_{\text{equ}} = \frac{AA'_0\beta'_0}{(1 + \omega^2\tau_M^2)(1 + \omega^2\tau_F^2)} \cdot \left| 1 - \omega^2\tau_M\tau_F - \frac{(1 + \omega^2\tau_M^2)(1 + \omega^2\tau_F^2)}{\beta'_0 A'_0} + j\omega(\tau_F + \tau_M) \right|. \tag{8}$$

The above equation, which yields both gain and phase-shift of the equivalent open loop as a function of the frequency, proves that the equivalent loop is « absolutely » stable, since the phase-shift cannot reach 180°. Therefore, the loop gain may be raised as long as other time constants do not invalidate eq. (8) (which does not happen in our case up to loop gains of about 300 using the d.c. error-signal channel). Finally, it is worthwhile to note that the asymptotic value of the gain as given by eq. (8), is A: the real high-frequency cut-off characteristic is determined by the parasitic suppressors (see Sect. 3·2).

The foregoing analysis exemplifies a typical situation which occurs when one uses the spectrometer in X-band. Anyway, the working conditions of the loop may be largely varied (and the above analysis may be easily extended to the different conditions), by varying, for instance, the number of the EL34 tubes; or the value of β; or feeding independently some of the sections of the magnet; or by drawing part of the current which energizes the magnet through a high wattage resistor which parallels the EL34's. The latter case would be a rather unusual one with our equipment, because of the large number of tubes available in the power stage. Anyway, since the present work has been completed, an apparatus using such a resistor across a small number of parallel-tubes, has been described [14].

[14] R. J. Abraham, D. W. Ovenall and D. H. Whiffen: *Journ. Sci. Inst.*, **34**, 269 (1957).

3·5. *Modulation and measurement of the magnetic field.* – It is very convenient to modulate the magnetic absorption signal via a modulation of the magnetic field which is small compared with the linewidth. Also, it may be convenient sometimes to use a field modulation as large as the width of the whole line or spectrum, in order to get a convenient oscillogram.

In our case the modulation may go up to 30 gauss peak-to-peak and is obtained using a pair of flat coils arranged in the gap in such a way that the modulation occurs almost only within the gap rather than in the massive iron core. The coils are transformer-coupled to a power amplifier which uses a bank of eight EL34 tubes in parallel, and is driven at 59 Hz by a very stable oscillator. The same oscillator provides the reference signal used for the phase-sensitive detection of the absorption signal (see Sect. 5). A small auxiliary coil and a V.T.V.M. provide monitoring and measurement of the magnetic field modulation.

An approximate measurement system of the magnetic field is given by the rotating coil and a Poggendorf bridge, similar to the one used in the a.c. error-signal channel (see Sect. 3·3).

In addition, a precision measurement is given by a proton-resonance signal provided by a small probe which is part of a circuit similar to the one described by Knoebel and Hahn ([15]). In our case it may be used either with the original frequency-modulation system, or by taking advantage of the magnetic field modulation. The proton-resonance frequency is measured by comparison with a secondary frequency standard. Since the proton-resonance oscillator is very stable, the system provides a very simple and accurate monitoring of the magnetic field stability and homogeneity, simply by the use of a very small field modulation. The effect due to « wiggles » in the resonance signal is, for our work, negligible.

4. – The microwave system.

4·1. *Design considerations.* – As we have seen in Sect. 1, the absorbed power (far from saturation) is proportional to H_{rf}^2. It follows that the sensitivity in the detection of magnetic absorption is increased putting the sample in a high-Q cavity. The absorption may be detected measuring the variations of the r.f. power (or amplitude) reflected or transmitted by the cavity. The use of a transmission or a reflection cavity yields almost identical sensitivities as it is shown, for instance, in the interesting paper by G. Feher ([6]) on the ultimate sensitivity. We have chosen the use of a reflection cavity, since for low temperature work it offers the advantages of less space requirements, less thermal dispersion and less gas-bubble noise.

When a reflection cavity is used, the magic-T technique is convenient. Let us connect for instance the cavity and a variable load respectively to the side-arms 1 and 2 of the magic-T. Let the r.f. power come from arm 4 (H-arm) and a matched detector be mounted on arm 3 (E-arm). When magnetic resonance occurs, a variation of χ'' gives rise to a variation of the reflected amplitude; a variation of χ' gives rise to a variation of the resonance frequency

([15]) H. W. Knoebel and E. L. Hahn: *Rev. Sci. Inst.*, **22**, 904 (1951).

of the cavity i.e. to a variation of the phase of the reflected wave, together with a second-order variation of its amplitude (if the resonant-frequency shift is small). If E_1 is the reflected amplitude from the cavity, if E_2 is the reflected amplitude from the variable load and if φ is the relative phase at the magic-T, the detector is reached by a power proportional to

$$E_d^2 = \frac{E_1^2}{2} + \frac{E_2^2}{2} - E_1 E_2 \cos\varphi \,, \tag{9}$$

whereas the reflected power toward the r.f. source is proportional to

$$E_r^2 = \frac{E_1^2}{2} + \frac{E_2^2}{2} + E_1 E_2 \cos\varphi \,. \tag{10}$$

If the effect of the magnetic resonance is to change E_1 by an amount $\mathrm{d}E^1$ and φ by an amount $\mathrm{d}\varphi$ (where $\mathrm{d}E_1$ and $\mathrm{d}\varphi$ are small, as we will assume if not otherwise specified), the power on the detector is now (to the first order) proportional to:

$$E_d'^2 = E_d^2 + (E_1 - E_2\cos\varphi)\,\mathrm{d}E_1 + E_1 E_2 \sin\varphi\,\mathrm{d}\varphi \tag{11}$$

and the reflected power toward the r.f. source is proportional to

$$E_r'^2 = E_r^2 + (E_1 + E_2\cos\varphi)\,\mathrm{d}E_1 - E_1 E_2 \sin\varphi\,\mathrm{d}\varphi \,. \tag{12}$$

Eqs. (11) and (12) show that the total change in the reflected power is proportional to $2E_1\mathrm{d}E_1$, i.e. it equals the change in the power reflected by the cavity, which equals the change in the absorbed power. But eq. (11) shows further that the variation of the power at the detector may be even much larger than the absorbed power (*). The series expansion of E_d', as given by eq. (11) shows that, provided that $|E_d'^2 - E_d^2| \ll E_d^2$, the response of either a linear or a square-law detector is always proportional to the power variation in the detector arm. It is such a variation that we want to maximize.

Let us now give a brief account of the most important situations which may occur using the above magic-T mounting.

a) If the load in arm 2 is matched, then $E_2 = 0$, and the variation of the power to the detector is proportional to $E_1\mathrm{d}E_1$, i.e. it is one half of the reflected power from the cavity. The linear or square-law detector response

(*) The system behaves as a phase-detector, since the signal to be detected is previously mixed with the coherent signal E_2. In this case the situation in the detector arm is illustrated by the following example. Consider two voltage sources of e.m.f. F and f respectively, and negligible internal resistances. If separately shorted on a resistance R, the two sources supply, respectively, powers equal to F^2/R and f^2/R, whereas if connected in series (or in opposition), they supply an extra power $2(fF/R)$ or $-2(fF/R)$. The simultaneous presence of the two sources in the circuit forces each of them to supply an extra power $\pm(fF/R)$.

is then proportional to χ'' to the first order. The same result holds strictly if a square-law detector is used and if the r.f. oscillator frequency and the resonance frequency of the cavity are made to coincide. In this case one maylock the r.f. oscillator to the resonance frequency of the cavity, using an A.F.C. system. This minimizes the effects of the cavity vibrations induced by the modulating field, which, being H-dependent and coherent with the modulating field, may give a shift in the base-line of the recorded spectrum ([6]). The optimum sensitivity is reached if the power reflected from the cavity is maximized. This is obtained by the use of a coupling iris which renders the V.S.W.R. equal to $2 + \sqrt{3}$ ([6,9]). It seems worthwhile to note, anyway, that the maximum is very broad and it does not necessarily coincide with an optimum signal-to-noise ratio, when a crystal detector is used, due to the anomalous dependence of the crystal noise on the r.f. power level.

b) If an amplitude unbalance is left, namely if in eq. (11) is $E_1 \neq E_2$ and $\varphi = 0$ or $\varphi = \pi$ the power variation at the detector is proportional to $(E_1 - E_2)\,\mathrm{d}E_1$ and the detector response is proportional to χ'' ([16,1]). In order to maximize the signal, one has to maximize both $\mathrm{d}E_1$ and $(E_1 + E_2)$. The maximum of the reflected amplitude occurs for $E_1 = 0$, but for this value an inversion on the signal sign occurs ([6]): it is thus convenient to have E_1 small. though larger than $\mathrm{d}E_1$ and E_2 large. But if a crystal detector is used, a compromise in the bucking has to be made ([6]). Eq. (11) shows that the same results hold even if some phase unbalance is left, provided that the $\mathrm{d}\varphi$ term is cancelled by keeping the cavity at resonance.

c) If a small phase-unbalance is left, such that $\cos \approx 1$ but $\varphi \gg \mathrm{d}\varphi$, the system is sensitive only to phase variations and its response is proportional ([16,1]) to χ'. Considerations similar to the ones of the above case, may be made concerning sensitivity.

d) If complete balance is used (aside from the bridge stability problem) the response is in general not simple but it depends on both χ' and χ'' ([17]).

From the above considerations, it follows that if one is interested in χ'', the optimum sensitivity is obtained in case *b*); furthermore cases *b*) and *c*) require the same microwave mounting, which is a very important feature if one is interested in both χ' and χ''. Conversely, also case *a*) may be sometimes convenient, due to the possible minimization of the effects of the cavity vibrations.

4·2. *The apparatus.* – The block-diagram of our microwave apparatus (for X- and K-bands) is shown in Fig. 5: measurements corresponding to cases *a*), *b*), *c*) and *d*), may be made without any change in the mounting. Usually, most measurements are performed in conditions *b*) and *c*).

The (medium power) klystron is immersed in a sealed, water-cooled oil bath, which provides proper shielding, cooling and temperature stability, and

([16]) N. Bloembergen, E. M. Purcell and R. V. Pound: *Phys. Rev.*, **73**, 679 (1948).

([17]) C. A. Whitmer, R. T. Weidner, J. S. Hsiang and P. R. Weiss: *Phys. Rev.*, **74**, 1478 (1948).

which preserves the possibility of manual adjustment of the klystron frequency. The highly stabilized power supply is suitable for high-voltage klystrons. The x-band klystron is followed by a ferrite isolator. The circuitry within the dotted Hewlett-Packard closed line in Fig. 5, which includes the magic-T No. 1, is used for the Pound I.F. automatic frequency control([21]) when measurements in conditions *b*) and *c*) are wanted. In these cases, it is more convenient to lock the frequency on a cavity other than the one which contains the sample. In fact, in so doing, magic-T No. 2 may be unbalanced as one wishes, using the variable probe which is in one of the side-arms.

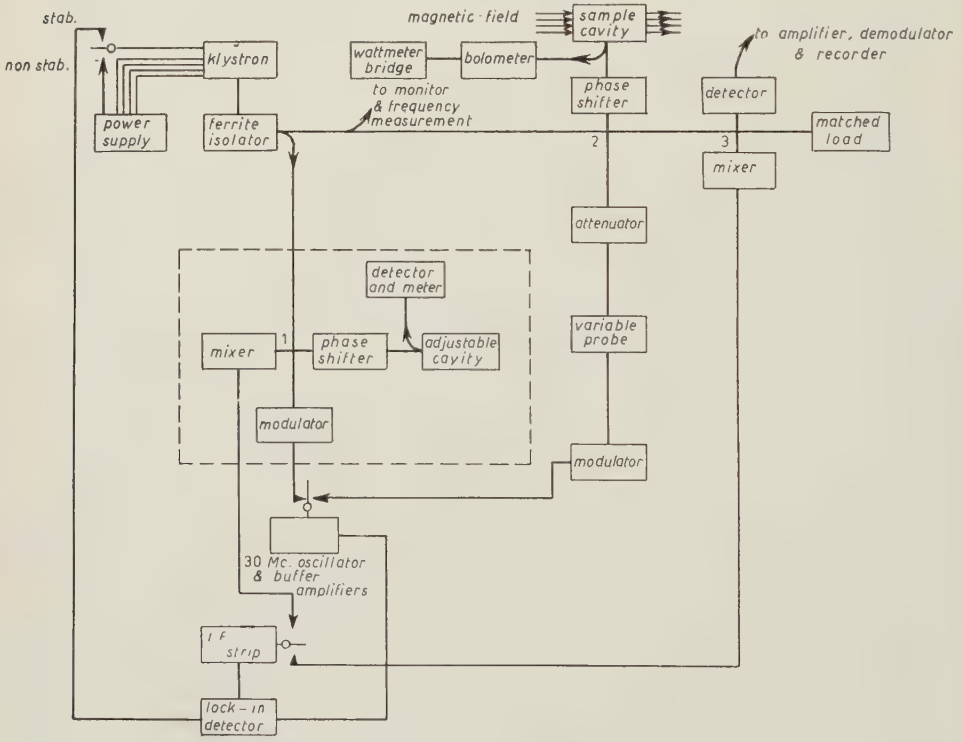

Fig. 5. – Block-diagram of the microwave instrumentation.

When condition *a*) has to be actuated, the above mentioned microwave circuitry (magic-T No. 1) and the variable probe in magic-T No. 2 are no longer used. Magic-T No. 3 serves as a power divider, and the klystron frequency is now locked on the sample cavity. In this case, the intermediate-frequency A.F.C. in the Zaffarano-Galloway mounting ([18]) has been found more convenient.

A sperry self-balancing barretter bridge is used for measurements of the power incident on (and reflected from) the sample cavity. The measurement is per-

([18]) F. P. Zaffarano and W. C. Galloway: Tech. Rep. No. 31, R.L.E., M.I.T., 1947.

formed using the directional coupler mounted on the sample cavity arm. A quantity proportional to the incident power may be measured by substituting the cavity by a short, or setting the cavity far from resonance. The measurement of a quantity proportional to the power reflected by the cavity at resonance allows the optimum sensitivity conditions to be reached; moreover, this measurement (or the one made on the directional coupler of magic-T No. 1) is convenient in order to lock the A.F.C. to the maximum of the cavity resonance ([9]).

The frequency being locked on a cavity, the temperature drifts limit the frequency stability. Anyway, the stability which may be reached (of the order of one part in 10^6) very largely fulfils our requirements. The frequency may be measured using a good secondary frequency standard. For most work in X-band, a good cavity wavemeter (having a precision better than 10^{-4}) is enough. However, a system comprehensive of a secondary frequency standard oscillator, frequency multiplier, mixer and receiver will be available in the near future.

Since the sensitivity is proportional to the unloaded Q of the cavity, high-Q cavities are used, which are similar to the one designed and used by the authors for measurements performed at M.I.T. ([19]), and described by STRANDBERG *et al.* ([9]). They are precision machined cylindrical cavities, using the TE_{011} mode. The sample holder, as shown in Fig. 6 lies orthogonal to the cylindrical axis, and passes through the center of the cavity. Fig. 6 shows also a groove which is used in order to lift the degeneracy with the TM_{111} mode ([20]). Since, at the center of the cavity, H^{rf} has a maximum and is directed along the axis of the cylinder, the methode descibed in Sect. 2 may be used. One extra advantage offered by this type of cavity, is given by the fact that the field modulation is in this case less effective in producing the mentioned vibrations.

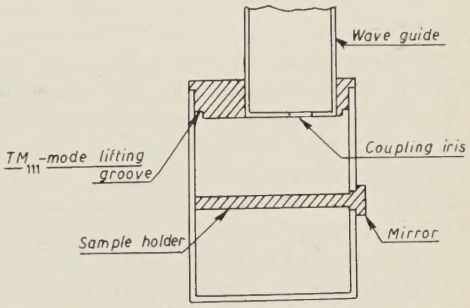

Fig. 6. – High-Q resonant cavity.

The absorption signal is given by the detector on magic-T No. 3. A crystal detector is usually used, but a bolometer may be used, particularly when absolute absorbed power measurements are required.

5. – Detection and recording of the magnetic absorption signal.

A modulate absorption signal is obtained using a magnetic field modulation which is accomplished as described in Sect. 3·5. The modulated absorption signal is injected through a well shielded step-up transformer into a conven-

([19]) M. B. PALMA-VITTORELLI, M. U. PALMA, D. PALUMBO and F. SGARLATA: *Nuovo Cimento*, **3**, 718 (1956).

([20]) C. G. MONTGOMERY (editor): *Technique of Microwave Measurements*, R.L.S., No. 11 (New York, 1948).

tional frequency-selective amplifier ([21]) having a few Hz bandwith and 135 db gain, similar to the one described by STRANDBERG *et al.* ([9]). The transformer makes the noise figure of the amplifier approach unity because at the same time it increases the effective impedance of the signal detector and the input signal amplitude. Its primary impedance is small, so that the short-circuit characteristics of the crystal detector hold.

The amplified signal undergoes phase-detection in a Brown converter driven by an amplified and phase-adjusted signal given by the same oscillator which drives the field-modulation power amplifier.

When a modulation is used which is small compared with the linewidth, the absorption signal is proportional to the derivative of the absorption curve. The search for the H-values for which the absorption maxima occur is made by a manual setting of the helical potentiometer which drives the magnetic field (see Sect. 3·3). The maximum condition is monitored by a central-zero meter at the output from the phase-detector. A plot of the whole spectrum is obtained by feeding the output from the phase-detector into a Speedomax recorder. The drive-shaft extension of the latter drives, through a gear, the shaft of the helical potentiometer which in turn drives the magnetic field. In so doing, linearity is preserved.

In order to have a good fidelity when recording a spectrum, the modulation amplitude has to be small. This causes a decrease of the sensitivity, by a factor roughly equal to the ratio between the modulation amplitude and the linewidth. If one is interested in the detection of the absorption only, the modulation amplitude may be raised up to the order of the linewidth. The sensitivity may also be raised using a larger time constant in the filter which follows the phase-detector, i.e. reducing the bandwith. Anyway, if a good fidelity is wanted, the recording time has to be in this case proportionally increased. A typical compromise is a recording time of the order of half an hour and a related overall bandwith which depends on the spectrum to be recorded.

The ultimate sensitivity has been given in Sect. 1.

* * *

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([21]) G. E. VALLEY and H. WALLMAN (editors): *Vacuum Tube Amplifiers*, R.L.S., No. 18 (New York, 1948).

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